FUNDAMENTALS OF CRYSTALLO-OPTICS,

acousto-optics, electro-optics

and magneto-optics

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Tensor and its transformation by rotation of coordinate system

Vector: $\mathbf{a} = a_x\mathbf{x}^0 + a_y\mathbf{y}^0 + a_z\mathbf{z}^0 = \sum_i a_i\mathbf{x}_i^0$, in „matrix“ representation $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

2-nd rank tensor: $\overline{\mathbf{T}} = \sum_i \sum_j T_{ij}\mathbf{x}_i^0\mathbf{x}_j^0$. Dyadic product of two vectors: $\mathbf{ab} = \sum_i \sum_j a_i b_j \mathbf{x}_i^0\mathbf{x}_j^0$

In matrixrepresentation $\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$.

3-nd rank tensor: $\tilde{\mathbf{r}} = \sum_i \sum_j \sum_k r_{ijk}\mathbf{x}_i^0\mathbf{x}_j^0\mathbf{x}_k^0$ 4-th rank tensor: $\overline{\overline{\mathbf{c}}} = \sum_i \sum_j \sum_k \sum_l c_{ijkl}\mathbf{x}_i^0\mathbf{x}_j^0\mathbf{x}_k^0\mathbf{x}_l^0$

Scalar products of tensors: $\mathbf{T}\cdot\mathbf{a} = \left(\sum_i \sum_j T_{ij}\mathbf{x}_i^0\mathbf{x}_j^0\right)\cdot\sum_k a_k\mathbf{x}_k^0 = \sum_i \sum_j \left(T_{ij}\mathbf{x}_i^0 \sum_k \mathbf{x}_j^0\cdot\mathbf{x}_k^0 a_k\right)$

$$= \sum_i \sum_j \left(T_{ij}\mathbf{x}_i^0 \sum_k \delta_{jk}a_k\right) = \sum_i \sum_j T_{ij}a_j\mathbf{x}_i^0 = \sum_i b_i\mathbf{x}_i^0,$$

$$b_i = \sum_j T_{ij}a_j$$

Tensor and its transformation by rotation of coordinate system - II

Double scalar product: $$\bar{\mathbf{T}}:\bar{\mathbf{S}} = \sum_i \sum_j \sum_k \sum_l T_{ij} S_{kl} \left[\mathbf{x}_i^0 \left(\mathbf{x}_j^0 \cdot \mathbf{x}_k^0 \right) \right] \cdot \mathbf{x}_l^0$$
$$= \sum_i \sum_j \sum_k \sum_l T_{ij} S_{kl} \delta_{il} \delta_{jk} = \sum_i \sum_j T_{ij} S_{ji}$$

Rotation of coordinates: original set $\mathbf{x}_1^0, \mathbf{x}_2^0, \mathbf{x}_3^0$, Rotated: $\mathbf{x}_1^{0\prime}, \mathbf{x}_2^{0\prime}, \mathbf{x}_3^{0\prime}$.

Matrix of directional cosines: $\alpha_{ij} = \mathbf{x}_i^{0\prime} \cdot \mathbf{x}_j^0 = \cos\left(x_i', x_j\right)$

Backward transformation: $\beta_{ji} = \mathbf{x}_j^0 \cdot \mathbf{x}_i^{0\prime} = \cos\left(x_j, x_i'\right) = \cos\left(x_i', x_j\right) = \alpha_{ij}$, $\boxed{\boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^T}$

Apparently $\mathbf{x}_i^{0\prime} = \sum_j \left(\mathbf{x}_i^{0\prime} \cdot \mathbf{x}_j^0 \right) \mathbf{x}_j^0 = \sum_j \alpha_{ij} \mathbf{x}_j^0, \quad \mathbf{x}_j^0 = \sum_i \beta_{ji} \mathbf{x}_i^{0\prime} = \sum_i \alpha_{ij} \mathbf{x}_i^{0\prime}$

Transformation of a vector: $\mathbf{a} = \sum_i a_i' \mathbf{x}_i^{0\prime} = \sum_j a_j \mathbf{x}_j^0 = \sum_j \sum_i \alpha_{ij} a_j \mathbf{x}_i^{0\prime}; \quad a_i' = \sum_j \alpha_{ij} a_j$

Analogously, $T_{ij}' = \sum_k \sum_l \alpha_{ik} \alpha_{jl} T_{kl}, \quad r_{ijk}' = \sum_l \sum_m \sum_n \alpha_{il} \alpha_{jm} \alpha_{kn} r_{lmn},$

$$c_{ijkl}' = \sum_m \sum_n \sum_p \sum_q \alpha_{im} \alpha_{jn} \alpha_{kp} \alpha_{lq} c_{mnpq}$$ etc.

Summation symbol is often omitted; if so, summation over repeated subscript(s) is supposed

Fundamentals of crystallo-optics

Optical wave propagation in an anisotropic medium

Harmonic time-dependent field in the medium without sources: $\rho = 0, \qquad \boldsymbol{J} = \mathbf{0}$

$$\mathtt{H}\,(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\right\} = \frac{1}{2}\left\{\mathbf{E}(\mathbf{r})e^{-i\omega t} + c.c.\right\}, \quad \mathtt{K}\;(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{H}(\mathbf{r})e^{-i\omega t}\right\} = \frac{1}{2}\left\{\mathbf{H}(\mathbf{r})e^{-i\omega t} + c.c.\right\}$$

Propagation is subject to Maxwell equations $\nabla \times \mathbf{E} = i\omega \mathbf{B}, \qquad \nabla \times \mathbf{H} = -i\omega \mathbf{D},$

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon} \cdot \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mathbf{H}\,.$$

The divergence equations are the direct consequence: $\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0.$

Anisotropy is determined by the relation between **E** a **D:** $\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon} \cdot \mathbf{E} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \cdot \mathbf{E}$

From general thermodynamics laws it follows that the relative permittivity tensor $\boldsymbol{\varepsilon}$ **in a lossless medium** is **hermitean,** $\overline{\varepsilon}^{T} = \overline{\varepsilon}^{*}$; we will mostly consider **real and symmetric** ε**.**

Real symmetric tensor $\boldsymbol{\varepsilon}$ can be **diagonalized** by suitable rotation of a coordinate system; in the new coordinate system the relative permittivity tensor $\boldsymbol{\varepsilon}$ is ***diagonal***

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix}.$$

From general properties of symmetric matrices it follows that its three eigenvectors (*permittivity* axes) are mutually orthogonal and the eigenvalues are real.

Classification of anisotropic media:

$\varepsilon_{xx} \neq \varepsilon_{yy} \neq \varepsilon_{zz},\ n_1 \neq n_2 \neq n_3$ ***optically biaxial*** medium (the most general; crystals)

$\varepsilon_{xx} = \varepsilon_{yy} \neq \varepsilon_{zz},\ n_1 = n_2 \neq n_3$ ***optically uniaxial*** medium (crystals, polymers, ...)

$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz},\ n_1 = n_2 = n_3$ ***isotropic*** medium (gases, most solids and liquids)

Optical plane wave propagation in an anisotropic medium

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_0 e^{ik_0\mathbf{l}\cdot\mathbf{r}}, \quad \mathbf{H} = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}} = \mathbf{H}_0 e^{ik_0\mathbf{l}\cdot\mathbf{r}}, \quad k_0 = \frac{2\pi}{\lambda}, \quad \mathbf{l} = \frac{\mathbf{k}}{k_0}, \quad \mathbf{l} = n\,\mathbf{l}^0$$

$\mathbf{l}\ldots$ dimensionless *normalized wave vector* and phase velocity: $\boldsymbol{v}_f = \frac{\omega}{|\mathbf{k}|}\mathbf{k}^0 = \frac{c}{|\mathbf{l}|}\mathbf{l}^0; \quad |\mathbf{l}| = n.$

Relations among field vectors

Analogously with the symbolics $\frac{\partial}{\partial t} \to -i\omega$ for plane waves $e^{ik_0 \mathbf{l}\cdot\mathbf{r}}$ it holds $\nabla \to ik_0\mathbf{l}$

Then $ik_0\mathbf{l}\times\mathbf{E}_0 = i\omega\mu_0\mathbf{H}_0, \quad ik_0\mathbf{l}\times\mathbf{H}_0 = -i\omega\mathbf{D}_0, \quad \mathbf{D}_0 = \varepsilon_0\overline{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0.$

From this $\mathbf{H}_0 = \frac{k_0}{\omega\mu_0}\mathbf{l}\times\mathbf{E}_0 = \frac{\sqrt{\mu_0\varepsilon_0}}{\mu_0}\mathbf{l}\times\mathbf{E}_0 = Y_0\,\mathbf{l}\times\mathbf{E}_0,\ Y_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}, \qquad Z_0 = Y_0^{-1} = \sqrt{\frac{\mu_0}{\varepsilon_0}}.$

$$\mathbf{D}_0 = -\frac{k_0}{\omega}\mathbf{l}\times\mathbf{H}_0 = -\frac{1}{c}\mathbf{l}\times\mathbf{H}_0, \qquad \mathbf{E}_0 = -\frac{k_0}{\omega\varepsilon_0}\overline{\boldsymbol{\varepsilon}}^{-1}\cdot(\mathbf{l}\times\mathbf{H}_0) = -Z_0\overline{\boldsymbol{\varepsilon}}^{-1}\cdot(\mathbf{l}\times\mathbf{H}_0).$$

Conclusions: 1. The vector triad $\mathbf{D}_0$, $\mathbf{H}_0$, $\mathbf{l}$ forms right-handed orthogonal set of vectors;

2. The vectors $\mathbf{E}_0$ and $\mathbf{H}_0$ are mutually orthogonal;

3. The vectors $\mathbf{D}_0$ and $\mathbf{E}_0$ are **not** generally mutually parallel;

4. The vectors $\mathbf{E}_0$, $\mathbf{D}_0$, $\mathbf{H}_0$ are all in-phase;

5. The direction of the Poynting vector is **not parallel** with the wave vector:

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_0\times\mathbf{H}_0^*\} = \frac{1}{2}\mathbf{E}_0\times\mathbf{H}_0 = \frac{1}{2}Y_0\mathbf{E}_0\times(\mathbf{l}\times\mathbf{E}_0) = \frac{1}{2}Y_0\left[\underbrace{\mathbf{l}(\mathbf{E}_0\cdot\mathbf{E}_0)}_{\text{parallel}} - \underbrace{\mathbf{E}_0(\mathbf{l}\cdot\mathbf{E}_0)}_{\text{not parallel}}\right].$$

$$|\mathbf{l}| = n.$$

„**Dispersion**“ (Fresnel) **equation** for anisotropic media:

$$ik_0\,\mathbf{l}\times\mathbf{H}_0 = ik_0\,\mathbf{l}\times\left(Y_0\,\mathbf{l}\times\mathbf{E}_0\right) = -i\omega\varepsilon_0\bar{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0, \quad \text{or} \quad \mathbf{l}\times\left(\mathbf{l}\times\mathbf{E}_0\right)+\bar{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0 = \mathbf{0}.$$

This can be rewritten into the form

$$\bar{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0+\mathbf{l}\left(\mathbf{l}\cdot\mathbf{E}_0\right)-\left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{E}_0 = \mathbf{0}, \qquad \text{or} \quad \boxed{\left(\bar{\boldsymbol{\varepsilon}}+\mathbf{l}\mathbf{l}-\mathbf{l}^2\mathbf{I}\right)\cdot\mathbf{E}_0 = \mathbf{0},}$$

where $\mathbf{a}\,\mathbf{b} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}$ is the dyadic product of vectors **a, b.**

The explicit form of the Fresnel equation is

$$\begin{pmatrix} \varepsilon_{xx}-l_y^2-l_z^2 & \varepsilon_{xy}+l_x l_y & \varepsilon_{xz}+l_x l_z \\ \varepsilon_{xy}+l_x l_y & \varepsilon_{yy}-l_x^2-l_z^2 & \varepsilon_{yz}+l_y l_z \\ \varepsilon_{xz}+l_x l_z & \varepsilon_{yz}+l_y l_z & \varepsilon_{zz}-l_x^2-l_y^2 \end{pmatrix}\cdot\begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A non-trivial solution $\mathbf{E}_0$ requires that the determinant of the system of equations is zero:

$$\Phi(\omega,\mathbf{l}) = \det\left(\bar{\boldsymbol{\varepsilon}}+\mathbf{l}\mathbf{l}-\mathbf{l}^2\mathbf{I}\right) = 0.$$

In the permittivity axes (in which the permittivity is diagonal) it sounds

$$\Phi(\omega,\mathbf{l}) = \left(\varepsilon_{xx}-l_y^2-l_z^2\right)\left(\varepsilon_{yy}-l_x^2-l_z^2\right)\left(\varepsilon_{zz}-l_x^2-l_y^2\right)+2l_x^2 l_y^2 l_z^2$$
$$-l_x^2 l_z^2\left(\varepsilon_{yy}-l_x^2-l_z^2\right)-l_y^2 l_z^2\left(\varepsilon_{xx}-l_y^2-l_z^2\right)-l_x^2 l_y^2\left(\varepsilon_{zz}-l_x^2-l_y^2\right) = 0.$$

After some manipulations we get

$$\Phi(\omega,\mathbf{l}) = \varepsilon_{xx} l_x^4 + \varepsilon_{yy} l_y^4 + \varepsilon_{zz} l_z^4 + \varepsilon_{xx} l_x^2 \left(l_y^2 + l_z^2\right) + \varepsilon_{yy} l_y^2 \left(l_x^2 + l_z^2\right) + \varepsilon_{zz} l_z^2 \left(l_x^2 + l_y^2\right)$$
$$- \varepsilon_{xx}\varepsilon_{yy}(l_x^2 + l_y^2) - \varepsilon_{xx}\varepsilon_{zz}(l_x^2 + l_z^2) - \varepsilon_{yy}\varepsilon_{zz}(l_y^2 + l_z^2) + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}.$$

$\Phi(\omega,\mathbf{l})$ is thus a **polynomial of the 4-th degree** in all variables l_x, l_y, l_z , **symmetric** with respect to inversion. For any pair l_x, l_y, the solution gives 2 values of $l_{z1,2}$ and 2 values of $l_{z3,4} = -l_{z1,2}$.

The surface $\Phi(\omega,\mathbf{l}) = 0$ is thus of the **4-th degree.** It is called a **wave vector surface**, sometimes also a **slowness surface** (the phase velocity is proportional to $1/|\mathbf{l}|$).

We will show that the **direction of power flow is orthogonal to the wave vector surface**.

The direction of power flow is given by the **group velocity** $\mathbf{v}_g = \frac{1}{k_0}\nabla_1 \omega, \left(v_{gx} = \frac{1}{k_0}\frac{\partial \omega}{\partial l_x}\right)$ etc.

Since $\frac{\partial \Phi}{\partial \omega} d\omega + \nabla_1 \Phi \cdot d\mathbf{l} = 0, \quad d\omega = -\left(\frac{\partial \Phi}{\partial \omega}\right)^{-1} \nabla_1 \Phi \cdot d\mathbf{l}, \quad \nabla_1 \omega = -\left(\frac{\partial \Phi}{\partial \omega}\right)^{-1} \nabla_1 \Phi,$ and thus

$\mathbf{v}_g = -\frac{1}{k_0}\left(\frac{\partial \Phi}{\partial \omega}\right)^{-1} \nabla_1 \Phi.$ The direction of power flow is thus parallel to the normal to the wave vector surface.

Since the power flow is given by the Poynting vector $\mathbf{S} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E} \times \mathbf{H}^*\right\}$,

both $\mathbf{E}$ and $\mathbf{H}$ are tangent to the surface, and $\mathbf{H}$ is also perpendicular to $\mathbf{l}$.

Alternative approach to anisotropic medium:
index-ellipsoid

$$\mathbf{l}\cdot\bar{\boldsymbol{\varepsilon}}^{-1}\cdot\mathbf{l} = 1, \quad \frac{l_x^2}{\varepsilon_{xx}} + \frac{l_y^2}{\varepsilon_{yy}} + \frac{l_z^2}{\varepsilon_{zz}} = 1$$

Let us introduce a **projector** in the wave vector space into the plane perpendicular to $\mathbf{l}$:

$$\mathbf{P} = \mathbf{I} - \mathbf{l}^0\,\mathbf{l}^0,$$ where $\mathbf{I}$ is a unit matrix.

Explicitly, in components,

$$\mathbf{P} = \begin{pmatrix} 1-(l_x^0)^2 & -l_x^0 l_y^0 & -l_x^0 l_z^0 \\ -l_y^0 l_x^0 & 1-(l_y^0)^2 & -l_y^0 l_z^0 \\ -l_z^0 l_x^0 & -l_z^0 l_y^0 & 1-(l_z^0)^2 \end{pmatrix}$$

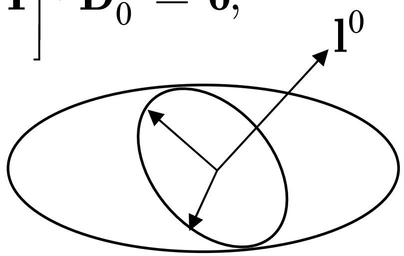
Since $\mathbf{D}_0$ is perpendicular to $\mathbf{l}$, the projection does not change $\mathbf{D}_0$:

$$\mathbf{P}\cdot\mathbf{D}_0 = (\mathbf{I} - \mathbf{l}^0\,\mathbf{l}^0)\cdot\mathbf{D}_0 = \mathbf{D}_0.$$

Then equation $\mathbf{l}\times\left(\mathbf{l}\times\mathbf{E}_0\right) + \bar{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0 = \mathbf{0}$ can be rewritten as $\left(\mathbf{l} = n\mathbf{l}^0,\ \mathbf{E}_0 = \dfrac{1}{\varepsilon_0}\bar{\boldsymbol{\varepsilon}}^{-1}\cdot\mathbf{D}_0\right)$

$$n^2\mathbf{l}^0\times\left(\mathbf{l}^0\times\frac{1}{\varepsilon_0}\bar{\boldsymbol{\varepsilon}}^{-1}\cdot\mathbf{D}_0\right) + \frac{1}{\varepsilon_0}\mathbf{D}_0 = \mathbf{0},$$ or $$\left[\left(\mathbf{I} - \mathbf{l}^0\,\mathbf{l}^0\right)\cdot\bar{\boldsymbol{\varepsilon}}^{-1} - \frac{1}{n^2}\mathbf{I}\right]\cdot\mathbf{D}_0 = \mathbf{0},$$

$$\left[\mathbf{P}\cdot\bar{\boldsymbol{\varepsilon}}^{-1} - \frac{1}{n^2}\mathbf{I}\right]\cdot\mathbf{P}\cdot\mathbf{D}_0 = \mathbf{0}, \qquad \boxed{\left[\mathbf{P}\cdot\bar{\boldsymbol{\varepsilon}}^{-1}\cdot\mathbf{P} - \frac{1}{n^2}\mathbf{I}\right]\cdot\mathbf{D}_0 = \mathbf{0}.}$$



$\mathbf{P}\cdot\bar{\boldsymbol{\varepsilon}}^{-1}\cdot\mathbf{P}$ is, in fact, a 2D tensor in the plane perpendicular to $\mathbf{l}$, and the red-framed equation is the equation of an ellipse in the plane of the components of $\mathbf{D}_0$. Hence the construction of the index-ellipsoid and the orientation of $\mathbf{D}_0$.

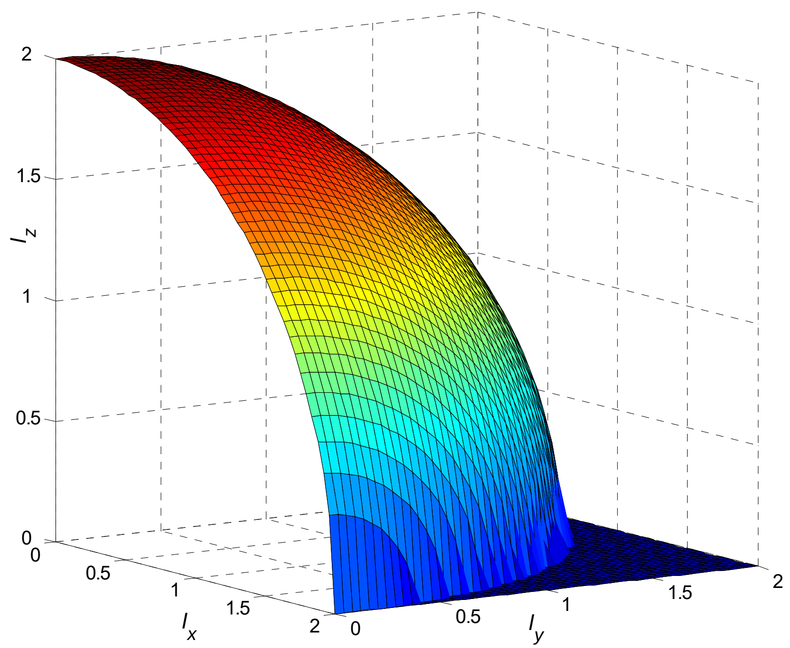
The general dispersion equation

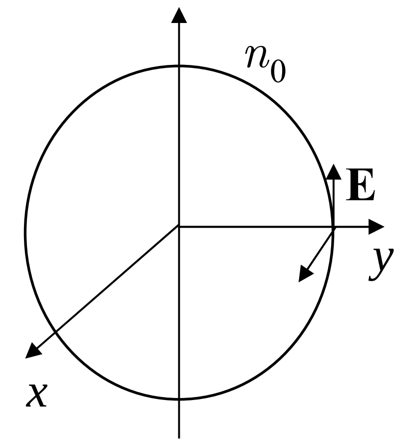
$$\Phi(\omega,\mathbf{l}) = \varepsilon_{xx} l_x^4 + \varepsilon_{yy} l_y^4 + \varepsilon_{zz} l_z^4 + \varepsilon_{xx} l_x^2 \left(l_y^2 + l_z^2\right) + \varepsilon_{yy} l_y^2 \left(l_x^2 + l_z^2\right) + \varepsilon_{zz} l_z^2 \left(l_x^2 + l_y^2\right) - \\ - \varepsilon_{xx}\varepsilon_{yy}(l_x^2 + l_y^2) - \varepsilon_{xx}\varepsilon_{zz}(l_x^2 + l_z^2) - \varepsilon_{yy}\varepsilon_{zz}(l_y^2 + l_z^2) + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} = 0$$

can be specified for the following specific cases:

In an isotropic medium it is reduced to the equation of a doubly degenerate sphere:

$$\Phi(\omega,\mathbf{l}) = \varepsilon_{xx}\left(\varepsilon_{xx} - l_x^2 - l_y^2 - l_z^2\right)^2 = 0.$$





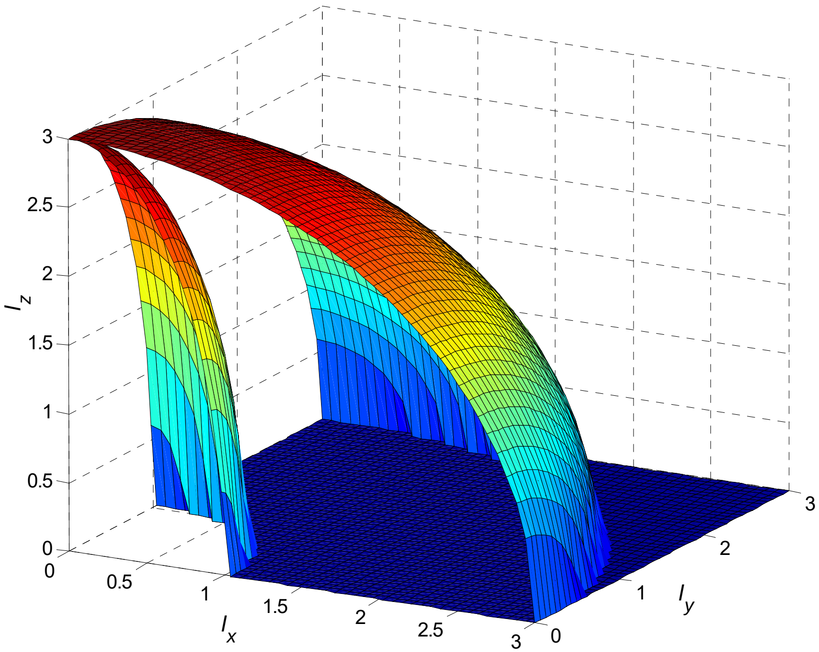
A uniaxial medium: $\Phi(\omega,\mathbf{l}) = \left(\varepsilon_{xx} - l_x^2 - l_y^2 - l_z^2\right)\left[\varepsilon_{xx}\varepsilon_{zz} - \varepsilon_{xx}\left(l_x^2 + l_y^2\right) - \varepsilon_{zz} l_z^2\right] = 0.$

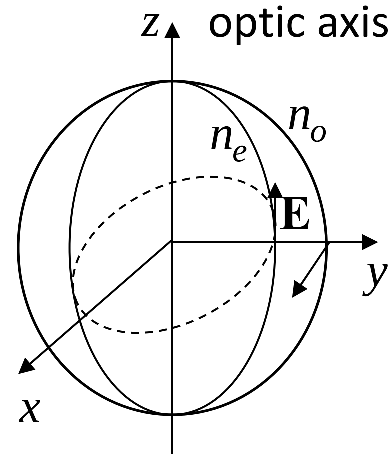
Equation of a sphere (ordinary wave): $\varepsilon_{xx} - l_x^2 - l_y^2 - l_z^2 = 0,$

Ellipsoid of rotation (extraordinary wave): $\dfrac{l_x^2 + l_y^2}{\varepsilon_{zz}} + \dfrac{l_z^2}{\varepsilon_{xx}} = 1,$ or $\dfrac{l_x^2 + l_y^2}{n_e^2} + \dfrac{l_z^2}{n_o^2} = 1$

In the x-z plane it holds $n = |\mathbf{l}| = l = \dfrac{n_x n_z}{\sqrt{n_x^2 \sin^2\theta + n_z^2 \cos^2\theta}},$

θ is a polar angle between the optic axis and $\mathbf{l}$.



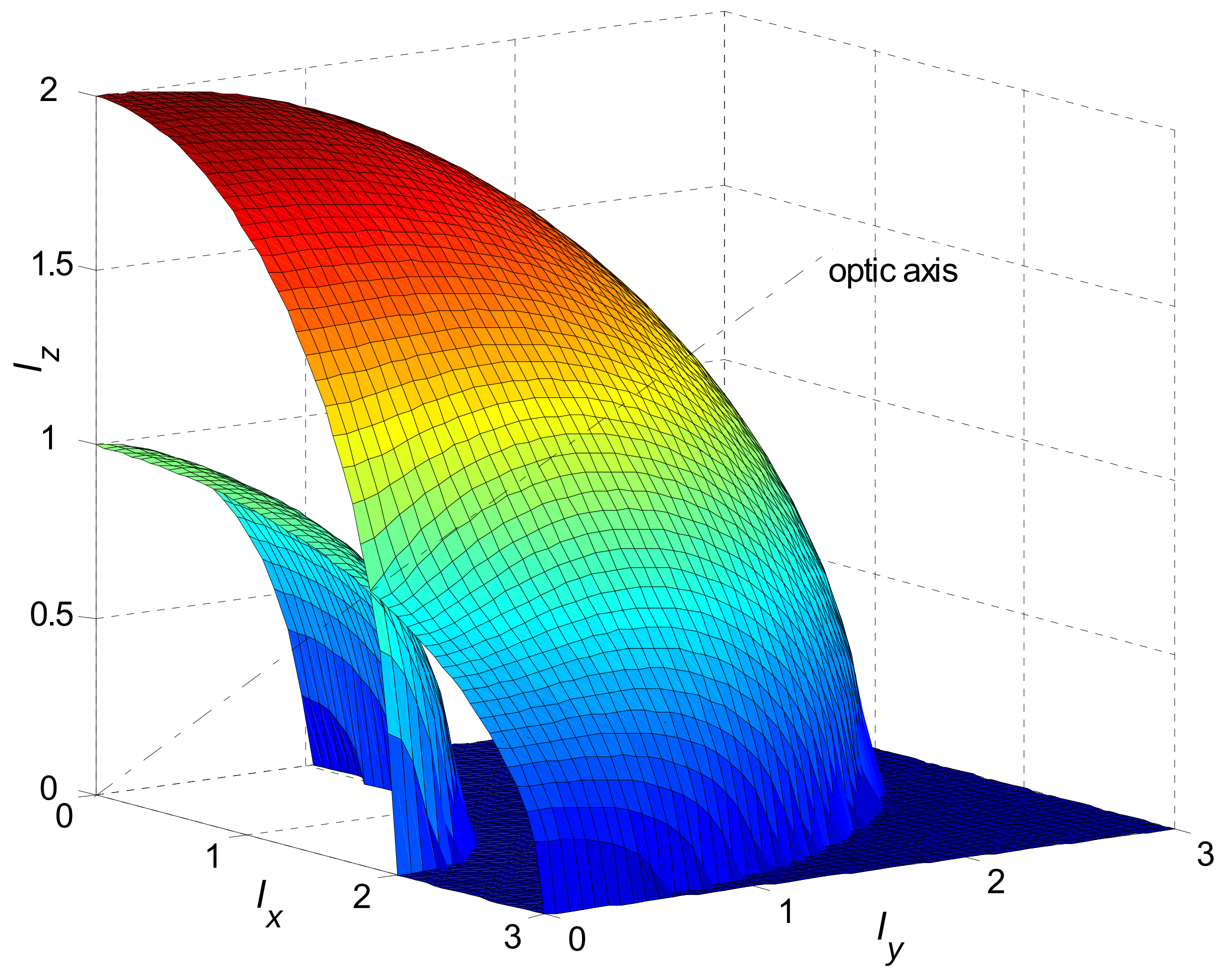


Biaxial medium

$$\Phi(\omega,\mathbf{l}) = \varepsilon_{xx} l_x^4 + \varepsilon_{yy} l_y^4 + \varepsilon_{zz} l_z^4 + \varepsilon_{xx} l_x^2 \left(l_y^2 + l_z^2\right) + \varepsilon_{yy} l_y^2 \left(l_x^2 + l_z^2\right) + \varepsilon_{zz} l_z^2 \left(l_x^2 + l_y^2\right)$$
$$- \varepsilon_{xx}\varepsilon_{yy}(l_x^2 + l_y^2) - \varepsilon_{xx}\varepsilon_{zz}(l_x^2 + l_z^2) - \varepsilon_{yy}\varepsilon_{zz}(l_y^2 + l_z^2) + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}$$

(cannot be simplified)

Wave vector surface of a biaxial medium with refractive indices $n_x = 1$, $n_y = 2$, $n_z = 3$



The cuts of the wave vector surface of a **biaxial medium** by the planes $l_x = 0,\ l_y = 0,\ l_z = 0$ can be expressed by the following forms:

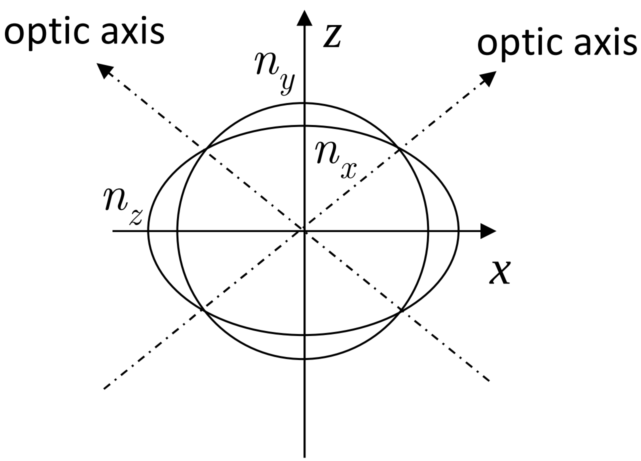
$$l_x = 0: \qquad \left(\varepsilon_{xx} - l_y^2 - l_z^2\right)\left[\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{yy} l_y^2 - \varepsilon_{zz} l_z^2\right] = 0,$$

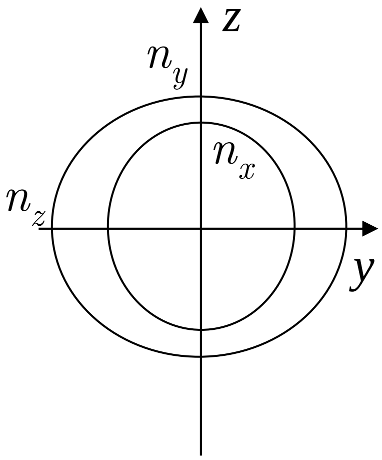
$$l_y = 0: \qquad \left(\varepsilon_{yy} - l_x^2 - l_z^2\right)\left[\varepsilon_{xx}\varepsilon_{zz} - \varepsilon_{xx} l_x^2 - \varepsilon_{zz} l_z^2\right] = 0,$$

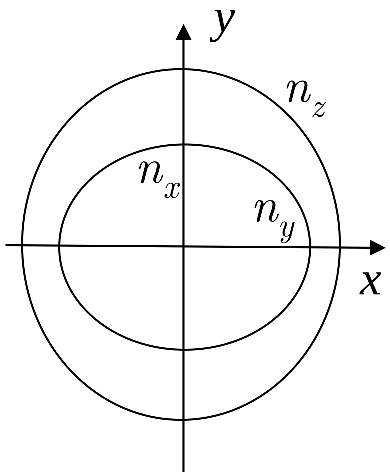
$$l_z = 0: \qquad \left(\varepsilon_{zz} - l_x^2 - l_y^2\right)\left[\varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xx} l_x^2 - \varepsilon_{yy} l_y^2\right] = 0,$$

which is always the product of the equation of **a circle and an ellipse.**

For $n_x < n_y < n_z$ we get the following diagrams:

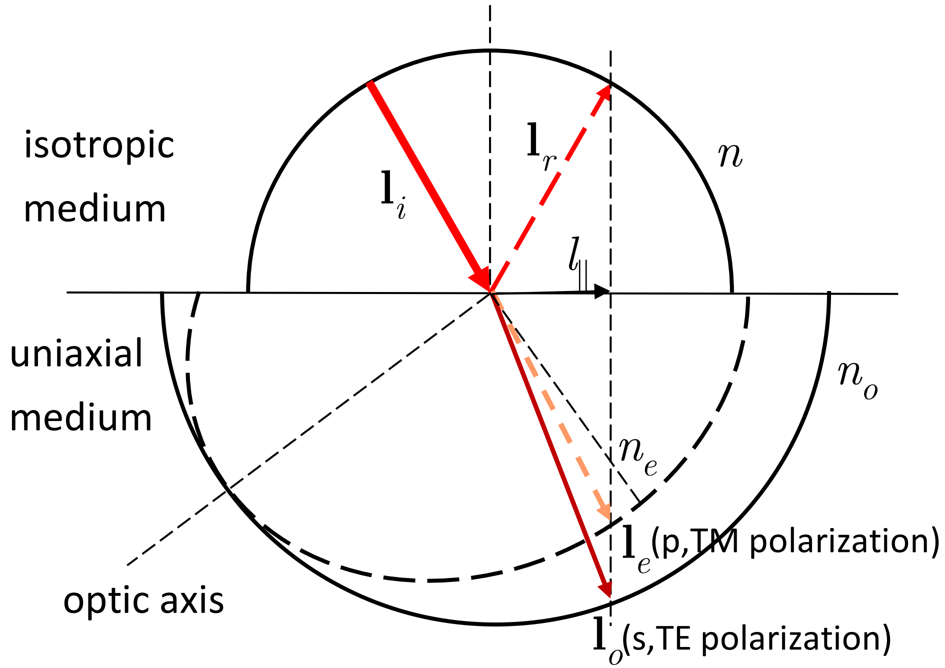






An example: incidence of a plane wave from an isotropic medium on a uniaxial medium

(A simple case – optic axis in the plane of incidence)



Generalized Fresnel coefficients of reflection and transmission can be derived from the continuity conditions of tangential $\mathbf{E}$ and $\mathbf{H}$ field components

Chiral (optically active) medium

Optical activity = rotation of a plane of polarization of a linearly polarized wave while propagating in a medium.

Chiral medium has no **translation symmetry.**

Constitutional (material) relations for chiral media can be defined in several ways. For plane waves they all reduce to

$$\mathbf{D} = \varepsilon_0 \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{E} - \frac{i}{c} \bar{\mathbf{g}} \cdot \mathbf{H} = \varepsilon_0 \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{E} - \frac{Y_0}{\omega} \bar{\mathbf{g}} \cdot \nabla \times \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \frac{i}{c} \bar{\mathbf{g}} \cdot \mathbf{E} = \mu_0 \mathbf{H} - \frac{Z_0}{\omega} \bar{\mathbf{g}} \cdot \bar{\boldsymbol{\varepsilon}}^{-1} \cdot \nabla \times \mathbf{H}$$

sign of a nonlocality

sign of a nonlocality

$\mathbf{g}$ is a dimensionless symmetric 2-nd rank tensor, called *chiral tensor*

Plane wave propagation in a chiral medium

Plane wave is described by $\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}},\ \mathbf{H} = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}},\ \mathbf{k} = k_0 \mathbf{l} = \omega\sqrt{\mu_0 \varepsilon_0}\,\mathbf{l} = \frac{\omega}{c}\mathbf{l} = \frac{2\pi}{\lambda}\mathbf{l}.$

The application of $\nabla \times$ results in

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} = ik_0 \mathbf{l} \times \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}},$$

$$\nabla \times \mathbf{H} = i\mathbf{k} \times \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}} = ik_0 \mathbf{l} \times \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}};$$

Substitution into Maxwell equations gives

$$\mathbf{l} \times \mathbf{E}_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H}_0 + i\,\mathbf{g} \cdot \mathbf{E}_0,$$

$$-\mathbf{l} \times \mathbf{H}_0 = \varepsilon\sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{E}_0 - i\,\mathbf{g} \cdot \mathbf{H}_0.$$

We get from the first equation $\mathbf{H}_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\mathbf{l}\times\mathbf{E}_0 - i\,\mathbf{g}\cdot\mathbf{E}_0\right).$

Introducing this into the second equation we get

$$-\mathbf{l}\times\left(\mathbf{l}\times\mathbf{E}_0 - i\,\overline{\mathbf{g}}\cdot\mathbf{E}_0\right) = \overline{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_0 - i\,\overline{\mathbf{g}}\cdot(\mathbf{l}\times\mathbf{E}_0 - i\,\overline{\mathbf{g}}\cdot\mathbf{E}_0)$$

We will again try to cast this equation into a set of three linear equations for $\mathbf{E}_0$:

$$[\mathbf{l}^2\,\mathbf{I} - \mathbf{l}\mathbf{l} + i\left(\mathbf{l}\times\overline{\mathbf{g}} + \overline{\mathbf{g}}\times\mathbf{l}\right) - \overline{\boldsymbol{\varepsilon}} + \cancel{\overline{\mathbf{g}}\cdot\overline{\mathbf{g}}}]\cdot\mathbf{E}_0 = \mathbf{0},$$

$$\mathbf{l}\times\overline{\mathbf{g}} = \sum_{m,n=1}^{3}(\mathbf{l}\times\mathbf{x}_m^0)\mathbf{x}_n^0 g_{mn},$$

Since $\left|g_{jk}\right| \ll 1$ for all known chiral media, we can neglect $\mathbf{g}\cdot\mathbf{g}$.

$$\overline{\mathbf{g}}\times\mathbf{l} = \sum_{m,n=1}^{3}\mathbf{x}_m^0(\mathbf{x}_n^0\times\mathbf{l})g_{mn}.$$

In the coordinate system in which ε is diagonal, the equation has the form

$$\begin{bmatrix} l_y^2 + l_z^2 - \varepsilon_{xx} & -l_x l_y + i\left[\left(g_{xx}+g_{yy}\right)l_z - g_{zx}l_x - g_{zy}l_y\right] & -l_x l_z - i\left[\left(g_{xx}+g_{zz}\right)l_y - g_{yx}l_x - g_{yz}l_z\right] \\ -l_x l_y - i\left[\left(g_{xx}+g_{yy}\right)l_z - g_{zx}l_x - g_{zy}l_y\right] & l_x^2 + l_z^2 - \varepsilon_{yy} & -l_y l_z - i\left[\left(g_{yy}+g_{zz}\right)l_x - g_{xy}l_y - g_{xz}l_z\right] \\ -l_x l_z + i\left[\left(g_{xx}+g_{zz}\right)l_y - g_{yx}l_x - g_{yz}l_z\right] & -l_y l_z + i\left[\left(g_{yy}+g_{zz}\right)l_x - g_{xy}l_y - g_{xz}l_z\right] & l_x^2 + l_y^2 - \varepsilon_{zz} \end{bmatrix}\cdot\begin{bmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{bmatrix} = \mathbf{0}.$$

This is the **Fresnel dispersion formula** for a plane wave in a chiral medium.

$\det[.] = 0$ … surface of the **4-th degree** in the $\left(l_x, l_y, l_z\right)$ space.

Isotropic chiral medium

$\overline{\mathbf{g}} = g\overline{\mathbf{I}},\ \ \overline{\boldsymbol{\varepsilon}} = \varepsilon\overline{\mathbf{I}};$ Let us choose $\mathbf{l} = l_z\mathbf{z}^0 = l_z\mathbf{x}_3^0.$

Then the dispersion formula is reduced into

$$\begin{pmatrix} l_z^2 - \varepsilon & -2i\,gl_z & 0 \\ 2i\,gl_z & l_z^2 - \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}.$$

The last equation has the solution $E_z = 0.$

The nontrivial solution of the first two requires that

$$\left(l_z^2 - \varepsilon\right)^2 - 4g^2l_z^2 = 0,$$

Since $g \ll \varepsilon,$

we immediately get

$$l_z = \pm\sqrt{\varepsilon + g^2} \pm g \approx \pm\sqrt{\varepsilon} \pm g = \pm n \pm g,$$

and for the field amplitudes we obtain

$$\frac{E_y}{E_x} \approx 2i\frac{gl_z}{l_z^2 - \varepsilon} = \pm i.$$

The “eigenwaves” of the isotropic chiral medium are thus circularly polarized waves which propagate with the refractive index of $n \pm g.$

Since all directions in an isotropic medium are equivalent, we can choose z in any direction. The wave vector surface consists of two concentric spheres with the radii $n \pm g.$

Rotation of the plane of polarization in an isotropic chiral medium

Let the electric field intensity at $z = 0$ is linearly polarized along $\mathbf{x}^0$: $\mathbf{E}(z=0) = E_0\mathbf{x}^0$.

At $z = 0$ it can be decomposed into two circularly polarized waves

$$\mathbf{E} = \frac{1}{2}\mathbf{E}_0^+ e^{ik_0 l^+ z} + \frac{1}{2}\mathbf{E}_0^- e^{ik_0 l^- z}, \quad \text{where} \quad \mathbf{E}_0^\pm = E_0(\mathbf{x}^0 \pm i\,\mathbf{y}^0), \;\; l^\pm = n \pm g.$$

Then

$$E_x(z) = \frac{1}{2}E_0\left(e^{ik_0 l^+ z} + e^{ik_0 l^- z}\right) = E_0 e^{ik_0(l^+ + l^-)z/2}\cos k_0\frac{\Delta l}{2}z,$$

$$E_y(z) = \frac{i}{2}E_0\left(e^{ik_0 l^+ z} - e^{ik_0 l^- z}\right) = -E_0 e^{ik_0(l^+ + l^-)z/2}\sin k_0\frac{\Delta l}{2}z.$$

$$\frac{\Delta l}{2} = \frac{l^+ - l^-}{2} = g$$

Propagating by the distance L results in the polarization rotation by $\varphi = k_0\frac{\Delta l}{2}L = k_0 g\,L$.

The chiral parameter *g* can be determined from the specific rotation power $\frac{\varphi}{L}$, $g = \frac{1}{k_0}\frac{\varphi}{L}$.

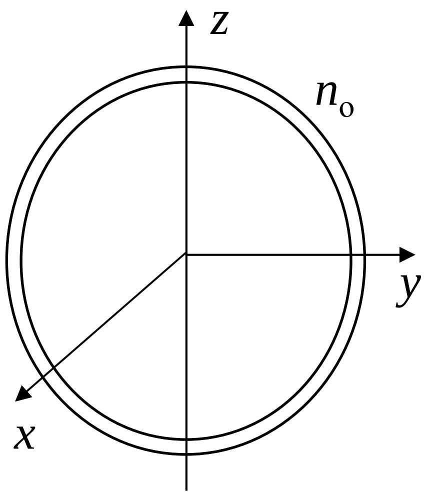
Specific rotation power and the chiral parameter of some materials at $\lambda = 632.8$ nm

material	φ/L	g
α−quartz SiO_2	22°/mm	3.85×10^{-5}
paratelurite TeO_2	87°/mm	1.52×10^{-4}
$Bi_{12}GeO_{20}$	20°/mm	3.5×10^{-5}

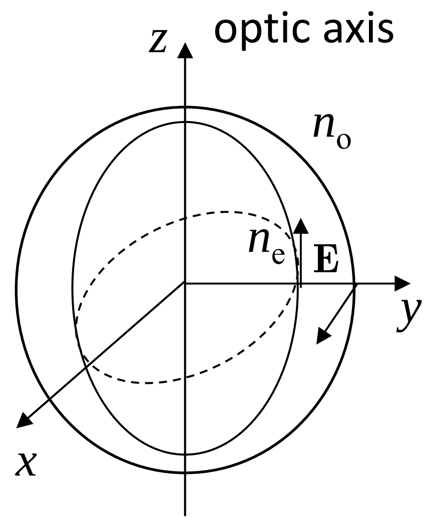
Note that $g \ll 1$, indeed.

The influence of chirality on the shape of wave vector surfaces

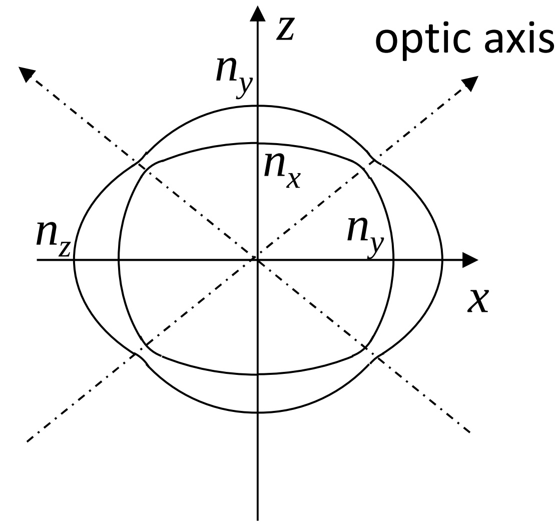
a) Isotropic medium



b) (Linearly) uniaxial medium



c) (Linearly) biaxial medium

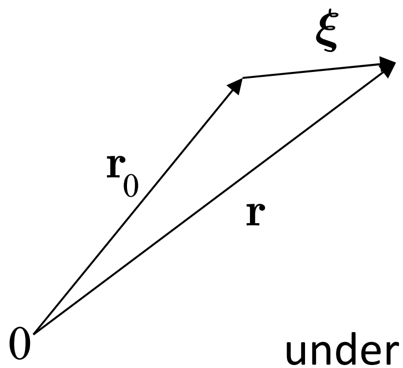


In the presence of the (linear) birefringence – uniaxial or biaxial –
the chirality is manifested only “in the vicinity” of optical axes (a few degree angular offset) otherwise it is overwhelmed by the (linear) birefringence.

Fundamentals of propagation of acoustic waves in elastic media

Propagation of acoustic waves in elastic media

(B.A.Auld: Acoustic fields and waves in solids I, II, J. Wiley 1973)



"Deformation" of a solid body: $\mathbf{r}\left(\mathbf{r}_0,t\right)=\mathbf{r}_0+\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)$

$\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)$ (elastic) deviation of the point $\mathbf{r}_0$ in time t

The distance between two closely spaced points $d\mathbf{r}_0$ is changed

under deformation to $\mathbf{r}_0+d\mathbf{r}_0+\boldsymbol{\xi}\left(\mathbf{r}_0+d\mathbf{r}_0,t\right)-\mathbf{r}_0-\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)=d\mathbf{r}_0+d\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)$,

where $d\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)=\sum_n \frac{\partial \boldsymbol{\xi}}{\partial x_n}dx_n=\sum_{m,n,p}\frac{\partial \xi_m}{\partial x_n}\mathbf{x}_m^0\underbrace{\mathbf{x}_n^0\cdot\mathbf{x}_p^0}_{\delta_{np}}dx_p=\nabla\boldsymbol{\xi}\left(\mathbf{r}_0,t\right)\cdot d\mathbf{r}_0$,

and $\nabla\sum_{m,n}\frac{\partial \xi_m}{\partial x_n}\mathbf{x}_m^0\mathbf{x}_n^0$ is the *gradient* of the deviation *vector* (dyadic).

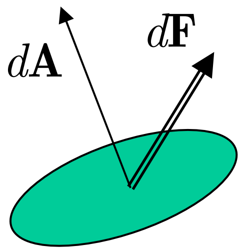
If $d\boldsymbol{\xi}\cdot d\mathbf{r}_0=0$, the size of $d\mathbf{r}_0$ is not changed. Then there is no *deformation* but only *rotation:*

$$d\boldsymbol{\xi}\cdot d\mathbf{r}_0=d\mathbf{r}_0.\nabla\boldsymbol{\xi}\cdot d\mathbf{r}_0=\sum_{m,n}\frac{\partial \xi_m}{\partial x_n}dx_m dx_n=\frac{1}{2}\sum_{m,n}\left(\frac{\partial \xi_m}{\partial x_n}+\frac{\partial \xi_n}{\partial x_m}\right)dx_m dx_n=0$$

The tensor of deformation is thus defined as a symmetric part of the tensor of the gradient of deviation:

$$\overline{\mathbf{S}}=\frac{1}{2}\left(\nabla\boldsymbol{\xi}+\left(\nabla\boldsymbol{\xi}\right)^T\right);\ S_{mn}=S_{nm}=\frac{1}{2}\left(\frac{\partial \xi_m}{dx_n}+\frac{\partial \xi_n}{dx_m}\right)$$

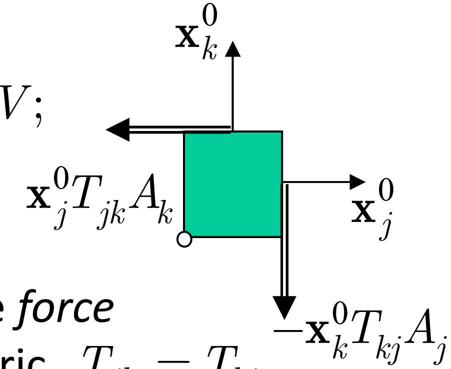
Force in solids



Force acting on a surface element ***dA*** is ***dF*:** $d\mathbf{F} = \bar{\mathbf{T}} \cdot d\mathbf{A}$

Force acting on a volume element dV:

$$d\mathbf{F} = \oiint_{dA} \bar{\mathbf{T}} \cdot d\mathbf{A} = \int_{dV} \nabla \cdot \bar{\mathbf{T}}\, dV = \sum_j \frac{\partial T_{jk}}{\partial x_k} \mathbf{x}_j^0\, dV;$$



$\bar{\mathbf{T}}$ … strain tensor

Since no volume element of solid in a steady state is rotating, the *force momentum must be zero;* as a result, the strain tensor is symmetric, $T_{jk} = T_{kj}$

For small $\left(\text{elastic}\right)$ deformations, there is a linear dependence between $\bar{\mathbf{T}}$ and $\bar{\mathbf{S}}$:

Generalized „Hook's law“: $T_{jk} = \sum_{lm} c_{jklm} S_{lm}$

From symmetries of $\bar{\mathbf{T}}$ and $\bar{\mathbf{S}}$ it follows $c_{jklm} = c_{kjml} = c_{jkml}$

Power density due to elastic deformation is

$$dU = \bar{\mathbf{T}} : d\bar{\mathbf{S}} = \sum_{jk} T_{jk} dS_{jk} = \sum_{jklm} c_{jklm} dS_{jk} S_{lm} = \sum_{jklm} c_{jklm} S_{jk} dS_{lm},$$ thus $c_{jklm} = c_{lmjk}$

Symmetry allows to introduce the shortened (Voigt) notation of tensor components:

$$c_{\alpha\beta} = c_{jklm},\ T_\alpha = T_{jk},\ S_\alpha = \begin{cases} S_{jj} \\ 2S_{jk},\ j \neq k \end{cases}$$

$$\alpha = 1, 2, \ldots, 6$$

Dynamics of (continuous) elastic media: acoustic waves

The "Newton's force equation" $\mathbf{F} = m \cdot \dfrac{d^2\mathbf{r}}{dt^2}$ for the volume element:

$$\frac{\partial^2}{\partial t^2}\int_V \rho\,\boldsymbol{\xi}\,dV = \oiint_A \bar{\mathbf{T}}\cdot d\mathbf{A} = \int_V \nabla\cdot\bar{\mathbf{T}}\,dV,$$ or $$\rho\frac{\partial^2\xi_j}{\partial t^2} = \sum_k \frac{\partial T_{jk}}{\partial x_k}.$$ Substituting for $\bar{\mathbf{T}}$

and considering the symmetry of $\bar{\mathbf{S}}$: $\rho\dfrac{\partial^2\xi_j}{\partial t^2} = \sum_{klm} c_{jklm}\dfrac{\partial^2\xi_m}{\partial x_k \partial x_l}\ldots$ wave equation for $\boldsymbol{\xi}$.

Acoustic plane wave: $\boldsymbol{\xi} = \boldsymbol{\xi}_0 e^{i(\mathbf{K}\cdot\mathbf{r}-\Omega t)}, \mathbf{K} = \dfrac{\Omega}{v_a}\mathbf{n}^0, \quad K = \dfrac{2\pi}{\Lambda}.$ The wave equation is then

$$\sum_m\left(\sum_{kl} c_{jklm} n_k n_l - \rho v_a^2 \delta_{jm}\right)\xi_{0m} = 0$$ …system of 3 linear equations for 3 components of $\boldsymbol{\xi}_0$.

Also: equation for eigenvalues ρv_a^2 and eigenvectors $\boldsymbol{\xi}_0$ of a real symmetric matrix

with elements $\sum_{kl} c_{jklm} n_k n_l$; there are 3 positive eigenvalues ρv_{aj}^2

and three real mutually orthogonal eigenvectors $\boldsymbol{\xi}_{0j}$ which define

the direction of vibration ("acoustic polarization") of the waves.

Since all components of $\boldsymbol{\xi}$ are real, the waves are linearly polarized.

Some properties of acoustic waves

It is possible to derive the following relation from energetic bilance of an acoustic wave:

$\mathbf{\Pi} = -\bar{\mathbf{T}} \cdot \dfrac{\partial \boldsymbol{\xi}}{\partial t}$. $\mathbf{\Pi}$ is an acoustic Poynting vector (density of power flow, [W/m^2]).

Group velocity $\mathbf{v}_g$ is parallel to $\mathbf{\Pi}$ and it satisfies a very interesting relation

$$\mathbf{v}_g \cdot \mathbf{n}^0 = v_a \quad \Rightarrow \quad \left|\mathbf{v}_g\right| \geq v_a \qquad (!!!)$$

In an isotropic medium it holds $c_{11} = c_{22} = c_{33},\ c_{12} = c_{13} = c_{23},$

Let's choose $\mathbf{n}^0 = \mathbf{z}^0$ for simplicity. We get $\quad c_{44} = c_{55} = c_{66} = \frac{1}{2}\left(c_{11} - c_{12}\right).$

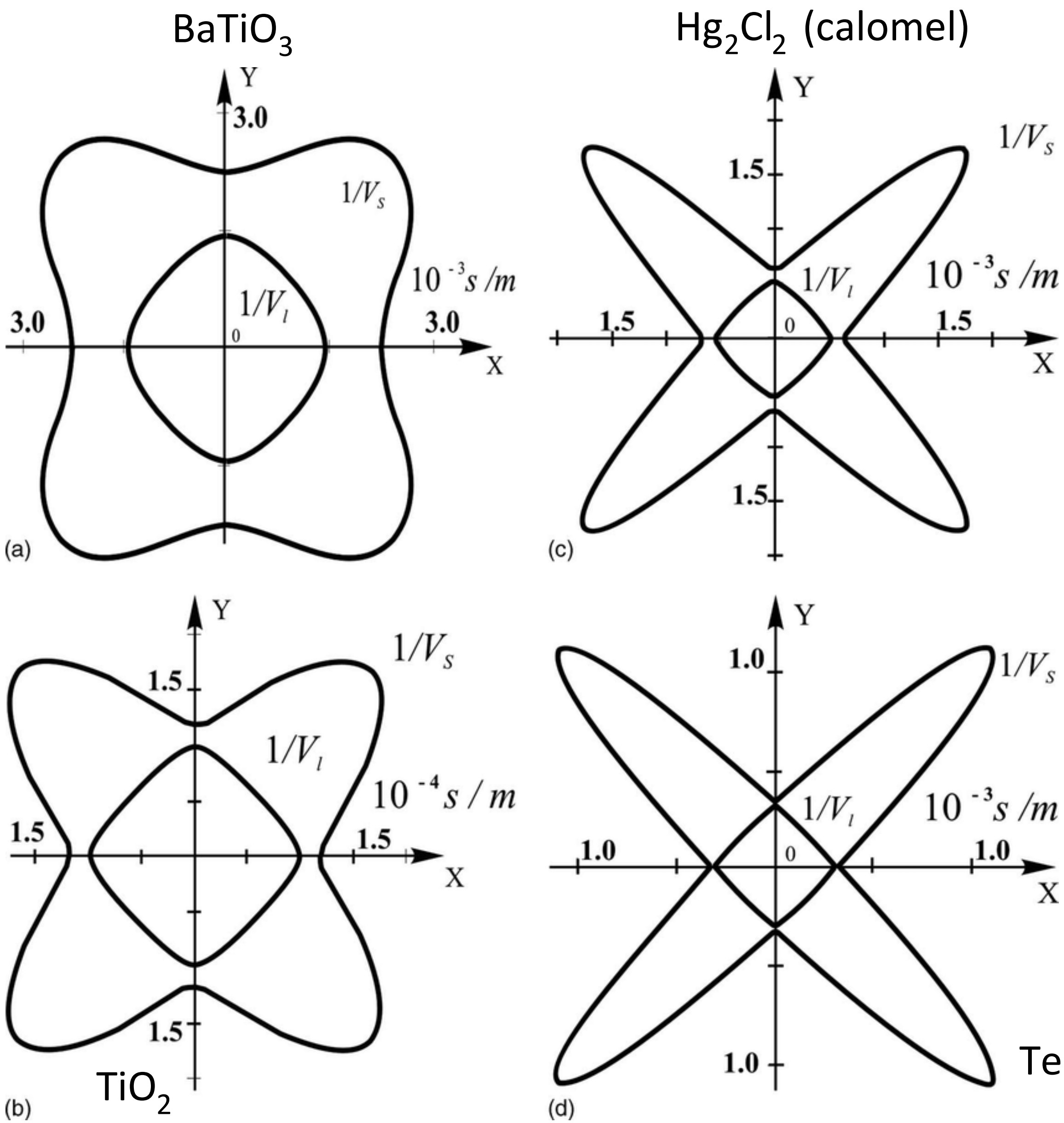
$$\begin{pmatrix} c_{44} - \rho v_a^2 & 0 & 0 \\ 0 & c_{44} - \rho v_a^2 & 0 \\ 0 & 0 & c_{11} - \rho v_a^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_{a\parallel} = \sqrt{\frac{c_{11}}{\rho}}, \qquad v_{a\perp} = \sqrt{\frac{c_{44}}{\rho}} < v_{a\parallel}$$

We introduce the normalized acoustic wave vector $\mathbf{l} = \mathbf{K}/\Omega$:

Then $\boldsymbol{\xi}_0 e^{i\Omega\left(\mathbf{l}_a \cdot \mathbf{r} - t\right)}$, and from the wave equation we get $\displaystyle\sum_m \left(\sum_{kl} c_{jklm} l_k l_l - \rho\,\delta_{jm}\right) \xi_{0m} = 0.$

$$\det\left[\left(\sum_{kl} c_{jklm} l_k l_l - \rho\,\delta_{jm}\right)\right] = 0$$ is the acoustic wave vector surface (of the 6-th degree!)

Cuts of acoustic wave vector surfaces of some crystals



The outer shells correspond to shear (transverse) acoustic waves that are in these crystal cuts degenerate.

The inner shell corresponds to longitudinal acoustic waves that are faster.

It is obvious that the acoustic anisotropy may be considerably larger than optical anisotropy of most materials

TeO_2 (paratellurite)

Theoretical background of acousto-optic interaction

Elastic deformation $\overline{\mathbf{S}}$ modifies the optical properties of the material; historically, this effect is expressed via the impermittivity tensor $\overline{\boldsymbol{\eta}} = \overline{\varepsilon}^{-1}$:

$\Delta\overline{\boldsymbol{\eta}} = \overline{\overline{\mathbf{p}}} : \overline{\mathbf{S}}$, where $\overline{\overline{\mathbf{p}}}$ is a tensor of photoelastic constants (photoelastic tensor).

For small deviations, $(\overline{\boldsymbol{\eta}} + \Delta\overline{\boldsymbol{\eta}}) \cdot (\overline{\varepsilon} + \Delta\overline{\varepsilon}) = \underbrace{\overline{\boldsymbol{\eta}} \cdot \overline{\varepsilon}}_{\overline{\mathbf{I}}} + \underbrace{\overline{\boldsymbol{\eta}} \cdot \Delta\overline{\varepsilon} + \Delta\overline{\boldsymbol{\eta}} \cdot \overline{\varepsilon}}_{\overline{0}} + \cancel{\Delta\overline{\boldsymbol{\eta}} \cdot \Delta\overline{\varepsilon}} = \overline{\mathbf{I}};$

thus
$$\boxed{\Delta\overline{\varepsilon} = -\overline{\varepsilon} \cdot \Delta\overline{\boldsymbol{\eta}} \cdot \overline{\varepsilon} = -\overline{\varepsilon} \cdot \overline{\overline{\mathbf{p}}} : \overline{\mathbf{S}} \cdot \overline{\varepsilon}}$$

Since both $\overline{\varepsilon}$ and $\overline{\mathbf{S}}$ are symmetric tensors of the 2-nd rank, $\overline{\overline{\mathbf{p}}}$ must be of rank 4, symmetric with respect to the exchange of the first and second pair of subscripts:

$$p_{ijkl} = p_{jikl} = p_{ijlk} = p_{jilk}.$$

A plane acoustic wave propagating in a medium has a vector of acoustic deviation $\boldsymbol{\xi}(\mathbf{r}, t)$:

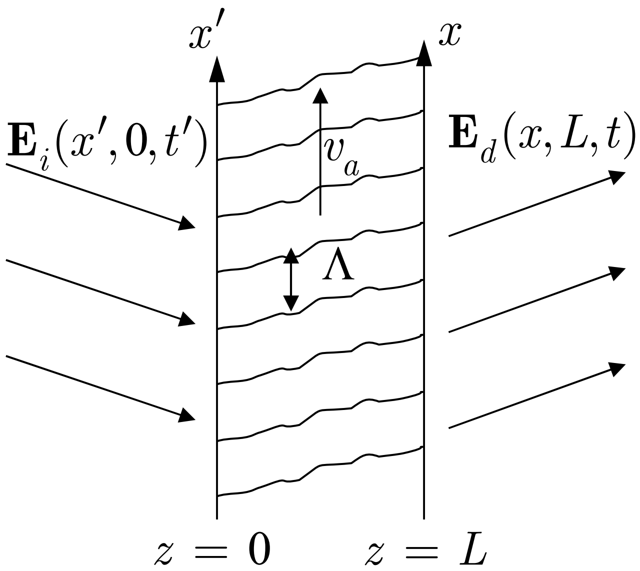
$$\overline{\boldsymbol{\xi}}(\mathbf{r}, t) = \mathrm{Re}\left\{\boldsymbol{\xi}_0 e^{i(\mathbf{K}\cdot\mathbf{r} - \Omega t)}\right\}; \ \overline{\mathbf{S}} = \frac{1}{2}\left[\nabla\boldsymbol{\xi} + (\nabla\boldsymbol{\xi})^T\right] = \frac{i}{2}\left(\mathbf{K}\boldsymbol{\xi}_0 + \boldsymbol{\xi}_0\mathbf{K}\right)e^{i(\mathbf{K}\cdot\mathbf{r} - \Omega t)},$$

$$\Delta\overline{\varepsilon}(\mathbf{r},t) = -\overline{\varepsilon} \cdot \overline{\overline{p}} : \left\{\frac{i}{2}\left[\left(\mathbf{K}\boldsymbol{\xi}_0 + \boldsymbol{\xi}_0\mathbf{K}\right)e^{i(\mathbf{K}\cdot\mathbf{r} - \Omega t)} + c.c.\right]\right\} \cdot \overline{\varepsilon} =$$

$$= \frac{\Omega}{2}\overline{\varepsilon} \cdot \overline{\overline{p}} : (\mathbf{l}\boldsymbol{\xi}_0 + \boldsymbol{\xi}_0\mathbf{l}) \cdot \overline{\varepsilon} \sin\left(\mathbf{K}\cdot\mathbf{r} - \Omega t\right) = \Delta\overline{\varepsilon} \sin\left(\mathbf{K}\cdot\mathbf{r} - \Omega t\right).$$

Modulation of permitivity due to an acoustic wave has thus the form of a *running plane wave*.

Diffraction of optical plane wave by running acoustic wave in isotropic medium



For weak intensities, the diffraction is a linear process. The transition of a wave through an acoustic column can be described by a generalized transfer function $\overline{\mathbf{F}}$:

$$\mathbf{E}_d(x,z=L,t)=\int_{-\infty}^{t}\int_{-\infty}^{\infty}\overline{\mathbf{F}}(x,x',t,t')\cdot\mathbf{E}_i(x',z=0,t')dx'dt'.$$

Since the acoustic wave is periodic both in space and time with *corelated* periods $\Lambda=2\pi v_a/\Omega$ and $T=2\pi/\Omega$, respectively, $\overline{\mathbf{F}}$ can be expanded in a Fourier series as follows:

$$\overline{\mathbf{F}}(x,x',t,t')=\sum_{q=-\infty}^{\infty}\overline{\mathbf{F}}_q(x-x',t-t')e^{iq(K_x x-\Omega t)}.$$

For the incident wave $\mathbf{E}_i(x',z=0,t')=\mathbf{E}_0e^{i(k_{ix}x'-\omega_i t')}$, the transmitted wave has the form

$$\mathbf{E}_d(x,z=L,t)=\sum_q\int_0^{\infty}\int_{-\infty}^{\infty}\overline{\mathbf{F}}_q(\xi,\tau)\cdot\mathbf{E}_0e^{-i(k_{ix}\xi-\omega_i\tau)}d\tau d\xi\, e^{i[(k_{ix}+qK_x)x-(\omega_i+q\Omega)t]}$$

$$=\sum_{q=-\infty}^{\infty}\mathbf{E}_qe^{i[(k_{ix}+qK_x)x-(\omega_i+q\Omega)t]};$$

The output consists of a superposition of plane waves with x-components of wave vectors $k_{d,qx}=k_{i,x}+qK_x$ and frequencies $\omega_q=\omega_i+q\Omega$ – *diffraction orders.*

Elasto-optic and photo-striction effects

In the previous slides we considered only the effect of an acoustic wave on the optical wave. However, a more rigorous analysis should take into account also the inverse effect.

The total change of internal energy of a unit volume of a medium under contemporary action of both electric field and elastic deformation is $dU = \mathbf{E}\cdot d\mathbf{D} + \bar{\mathbf{T}} : d\bar{\mathbf{S}}$.

Obviously, $\mathbf{D} = \varepsilon_0\left(\bar{\varepsilon} + \Delta\bar{\varepsilon}\right)\cdot\mathbf{E} = \varepsilon_0\bar{\varepsilon}\cdot\mathbf{E} - \underbrace{\varepsilon_0\bar{\varepsilon}\cdot\bar{\bar{\mathbf{p}}} : \bar{\mathbf{S}}\cdot\bar{\varepsilon}\cdot\mathbf{E}}_{elasto-optic\ contribution}$.

Let us introduce a new thermodynamic potential $V = U - \mathbf{E}\cdot\mathbf{D}$.

Then $dV = -\mathbf{D}\cdot d\mathbf{E} + \bar{\mathbf{T}} : d\bar{\mathbf{S}}$. The independent variables of V are thus $\mathbf{E}$ and $\bar{\mathbf{S}}$.

The equality of mixed derivatives gives $\frac{\partial V}{\partial E_j \partial S_{lm}} = -\frac{\partial D_j}{\partial S_{lm}} = \boxed{\varepsilon_0\varepsilon_{jr}\varepsilon_{ks}p_{rslm}E_k = \frac{\partial T_{lm}}{\partial E_j}}$;

here, the Einstein summation rules over j, k and s apply. Integrating the last equation we obtain the contribution of $\mathbf{E}$ to $\bar{\mathbf{T}}$, so that

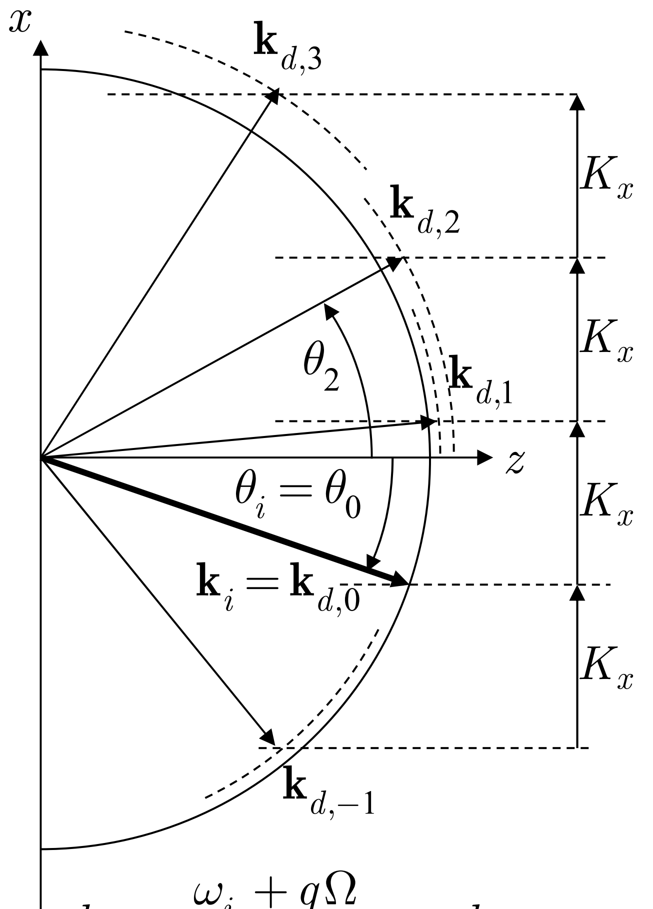
$$T_{rs} = \underbrace{c_{rslm}S_{lm}}_{\substack{elastic \\ deformation}} + \underbrace{\left(1/2\right)\varepsilon_0\varepsilon_{jl}\varepsilon_{km}p_{lmrs}E_jE_k}_{\substack{photostriction \\ (stimulated\ Brillouin \\ effect)}} .$$

For typical values encountered in technical acousto-optics,

$\varepsilon \approx 2,\ p \approx 0.2,\ c \approx 10^{10} \div 10^{11}$ N.m^{-2}, $S \approx 10^{-6},\ E \approx 10^{6}$ V.m^{-1},

the second term is typically by 5 orders of magnitude smaller and can thus be neglected.

Wave vector diagram for the diffraction by an acoustic wave in an isotropic medium



$$k_q = \frac{\omega_i + q\Omega}{c} n \approx k_i$$

Frequency shift of diffracted orders:

$$\omega_{d,q} = \omega_i + q\Omega \approx \omega_i$$

Typically,

$$\omega_i = \frac{2\pi c}{\lambda} \approx 2 \times 10^{15}\ \mathrm{s}^{-1} \quad (\lambda \approx 1\ \mu\mathrm{m})$$

$$\Omega = 2\pi f_a \approx 2\pi \times 10^8 \doteq 6 \times 10^8\ \mathrm{s}^{-1} \ll \omega_i$$

$(f_a \approx 100\ \mathrm{MHz})$

Wave vectors of diffracted waves:

$$k_{d,qx} = k_{ix} + qK_x,$$

$$k_{d,qz} = \sqrt{k_q^2 - \left(k_{ix} + qK_x\right)^2} \approx \sqrt{k_0^2 n^2 - \left(k_{ix} + qK_x\right)^2},$$

$$k_q = \frac{\omega_{d,q}}{c} n(\omega_{d,q}) \approx k_0 n.$$

Output angles of diffracted waves (grating equation):

$$\sin\theta_q \approx \sin\theta_0 + q\frac{K_x}{k_0 n} = \sin\theta_0 + q\frac{\lambda}{n\Lambda}.$$

$n \ldots$ the refractive index of the medium

Efficiency of acousto-optic interaction: the coupled-wave approach

Wave equation for the electric field intensity

$$\nabla\nabla\cdot\mathbf{E}-\Delta\mathbf{E}=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\overline{\varepsilon}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\right];$$

Let us choose for simplicity $\mathbf{E}(\mathbf{r},t)=\mathbf{y}^0E(x,z,t)$. Then ($\overline{\varepsilon}\cdot\mathbf{E}=n^2E$)

$$\frac{\partial^2}{\partial x^2}E(x,z,t)+\frac{\partial^2}{\partial z^2}E(x,z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\left[n^2(x,t)\right]E(x,z,t)\right\}=0 \quad (n \text{ does not depend on } z).$$

Assumptions of the coupled-wave theory:

$$E(x,z,t)\approx\sum_{q=-\infty}^{\infty}E_q(z)e^{i\left[\left(k_{ix}+qK\right)x+k_{q,z}z-\left(\omega_i+q\Omega\right)t\right]},\quad k_{q,z}\approx\sqrt{k_0^2n^2-\left(k_{ix}+qK\right)^2}.$$

$E_q(z)$ is a *slowly varying* complex amplitude, $\left|\frac{\partial^2E_q(z)}{\partial z^2}\right|\ll k^2\left|E_q(z)\right|,\ k\left|\frac{\partial E_q(z)}{\partial z}\right|,\ k=k_0n.$

$$n^2(x,t)\approx n^2+2nn_1\sin\left(Kx-\Omega t\right),\ n_1\approx\frac{\Delta\varepsilon}{2n}=-\frac{1}{2}n^3pS_0\ll n,$$

$$n(x,t)=\sqrt{n^2+\Delta\varepsilon\sin\left(Kx-\Omega t\right)}\approx n+n_1\sin\left(Kx-\Omega t\right).$$

Let us introduce new parameters

$$\Delta\varphi = \frac{k_0 n_1 L}{\cos\theta_i}, \quad Q = \frac{2\pi\lambda L}{n\Lambda^2\cos\theta_i}, \quad \alpha = -\frac{k_{ix}}{K} = -\frac{k}{K}\sin\theta_i = -\frac{n\Lambda}{\lambda}\sin\theta_i.$$

We insert the expansion of E into diffraction orders in the wave equation and make use of these parameters. Then we neglect all higher-order terms in n_1 and compare terms with identical exponents. As a result we get the following set of first-order equations:

$$\frac{\partial E_q(z)}{\partial z} = \frac{\Delta\varphi}{2L}\left(E_{q+1}(z) - E_{q-1}(z)\right) + \frac{iqQ}{2L}(2\alpha - q)E_q(z), \qquad q = 0, \pm 1, \pm 2, \dots$$

In a matrix form it sounds

$$\frac{d}{dz}\begin{pmatrix} \vdots \\ E_{-2} \\ E_{-1} \\ E_0 \\ E_1 \\ E_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} -i(2\alpha+2)\frac{Q}{2L} & \frac{\Delta\varphi}{2L} & 0 & \dots & \dots \\ -\frac{\Delta\varphi}{2L} & -i(2\alpha+1)\frac{Q}{2L} & \frac{\Delta\varphi}{2L} & 0 & \dots \\ 0 & -\frac{\Delta\varphi}{2L} & -i2\alpha\frac{Q}{2L} & \frac{\Delta\varphi}{2L} & 0 \\ \dots & 0 & -\frac{\Delta\varphi}{2L} & -i(2\alpha-1)\frac{Q}{2L} & \frac{\Delta\varphi}{2L} \\ \dots & \dots & 0 & -\frac{\Delta\varphi}{2L} & -i(2\alpha-2)\frac{Q}{2L} \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ E_{-2} \\ E_{-1} \\ E_0 \\ E_1 \\ E_2 \\ \vdots \end{pmatrix}$$

Note that in this approximation, only neighbouring orders are mutually coupled.

Raman-Nath and Bragg diffraction regimes

The observation that only neighbouring terms are mutually coupled is a consequence of a *purely sinusoidal* refractive index modulation.

The set of differential equations can be solved analytically in the two limit cases:

if $Q \ll 1$ or $Q \gg 1$.

1. $Q \ll 1$ – Raman–Nath regime
2. $Q \gg 1$ – Bragg regime.

1. $Q \ll 1$: **Raman - Nath regime.** In this case, the diagonal terms can be neglected:

$$\frac{\partial E_q(z)}{\partial z} = \frac{\Delta\varphi}{2L}\left(E_{q+1}(z) - E_{q-1}(z)\right), \qquad q = 0, \pm 1, \pm 2, \ \ldots$$

The initial condition is $E_q(0) = E_0 \delta_{q0}$ (only E_0 is nonzero at the input $z = 0$).

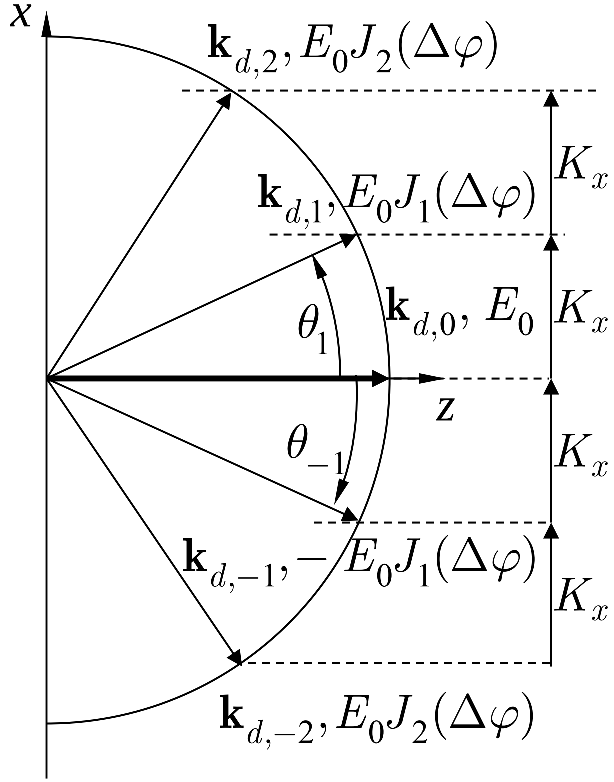
In this case, the analytic solution gives $E_q(L) = E_0 J_q(\Delta\varphi), \ \ q = 0, \pm 1, \pm 2, \ldots$

This effect has a very simple physical interpretation as a *phase modulation* of the incident wave by a column of acoustic wave (phase grating):

$$E(x, L, t) = E_0 e^{i\left(k_{ix} x - \omega_i t\right)} e^{i\Delta\varphi \sin\left(Kx - \Omega t\right)} = E_0 \sum_q J_q(\Delta\varphi) e^{i\left[(k_{ix} + qK)x - i\left(\omega_i + q\Omega\right)t\right]}.$$

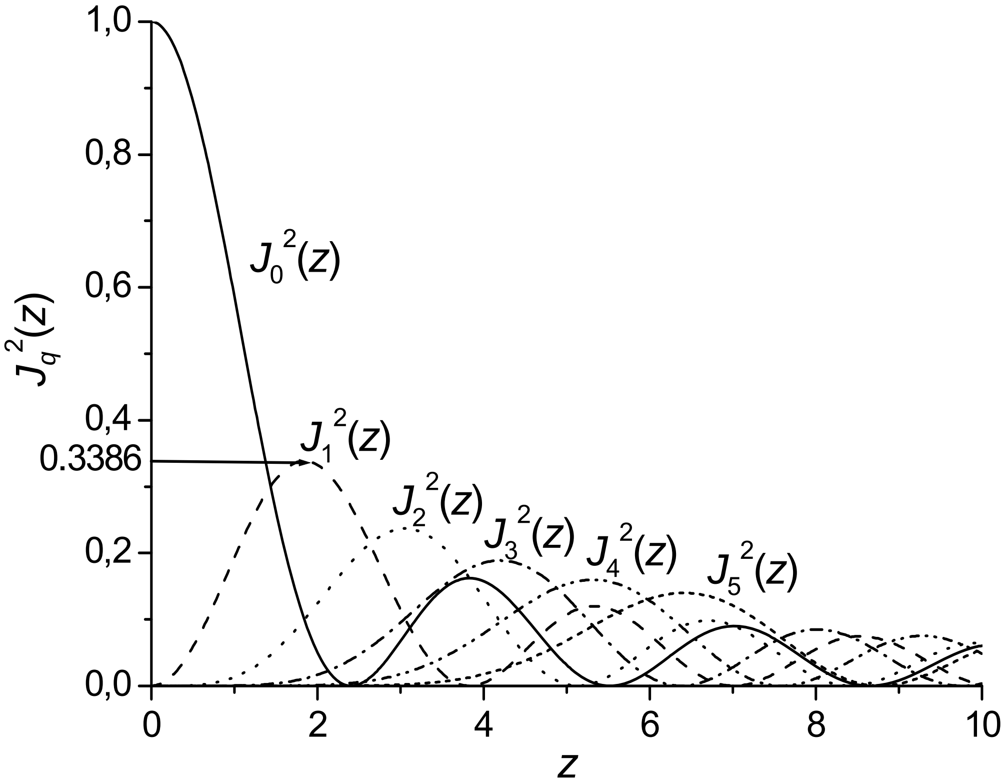
A more accurate solution for $0 < Q \ll 1, \ q \ll \alpha$ is $E_q(L) = E_0 J_q\left[\Delta\varphi \frac{\sin\left(Q\alpha/2\right)}{Q\alpha/2}\right]$.

Raman-Nath regime:



Diffraction goes into many diffraction orders; the diffraction efficiency of individual orders is given by squares of Bessel functions $J_q^2(\Delta\varphi)$.

The diffraction is strongly analogous to the diffraction by a thin phase (holographic) sinusoidal grating.



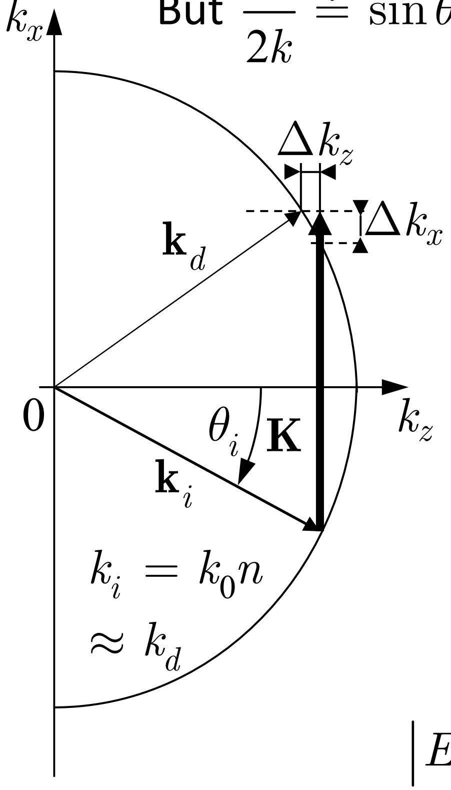
Bragg regime

Bragg regime takes place in theory for $Q \gg 1$, in reality for $Q \geq 10$.

Then it is possible to neglect the coupling with other orders except for $q \approx \pm 2\alpha$, typically the coupling is between orders $q = 0$ and 1 or -1. For $q = 1$

$$\frac{Q}{2}(1-2\alpha) = \frac{2\pi\lambda L}{2n\Lambda^2 \cos\theta_i}\left(1 + 2\frac{n\Lambda}{\lambda}\sin\theta_i\right) = \frac{KL}{2k\cos\theta_i}\left(K + 2k\sin\theta_i\right).$$

But $\frac{K}{2k} \doteq \sin\theta_i$, $K + 2k\sin\theta_i \doteq \Delta k_x$, and $\frac{Q}{2}(1-2\alpha) \doteq L\Delta k_x \tan\theta_i = \Delta k_z L$.



The set of coupled equations then reduces to two coupled equations

$$\frac{dE_0}{dz} = \frac{\Delta\varphi}{2L}E_1,$$
$$\frac{dE_1}{dz} = -\frac{\Delta\varphi}{2L}E_{d,0} + i\frac{\Delta k_z}{2L}E_1;$$

For the initial conditions $E_0(0) = E_0$, $E_1(0) = 0$ the solution is

$$E_1(L) = E_0\frac{\Delta\varphi}{2\sigma}e^{-i\frac{\Delta k_z L}{2}}\sin\sigma, \text{ where } \sigma = \sqrt{\left(\frac{\Delta k_z}{2}L\right)^2 + \left(\frac{\Delta\varphi}{2}\right)^2}.$$

$$\left|E_1(L)\right|^2 = E_0^2\left(\frac{\Delta\varphi}{2\sigma}\right)^2\sin^2\sigma, \text{ and for } \Delta k_z = 0,\ \left|E_1(L)\right|^2 = E_0^2\sin^2\left(\frac{\Delta\varphi}{2}\right).$$

Diffraction efficiency in the Bragg regime

Diffraction efficiency

$$\eta = \left|\frac{E_1(L)}{E_0(0)}\right|^2 = \left(\frac{\Delta\varphi}{2\sigma}\right)^2 \sin^2\sigma$$

Phase synchronism

$\Delta k_z L \approx 0$, i.e. $\mathbf{k}_i \pm \mathbf{K} = \mathbf{k}_d$

(conservation of (quasi)-momentum).

It holds $\frac{\Delta\varphi}{2} = \frac{2\pi n_1}{2\lambda\cos\theta_i} L \approx \frac{\pi}{2\lambda} n^3 p S_0 L.\cdot$ S_0 can be expressed as $S_0 = \sqrt{\frac{2\Pi_a}{\rho v_a^3}}$,

where $\Pi_a = \frac{1}{2}\rho v_a^3 S_0^2\ [\mathrm{W.m^{-2}}]$ is the acoustic power density (Poynting vector).

$$\frac{\Delta\varphi}{2} = \frac{\pi}{2\lambda} n^3 p S_0 L = \sqrt{\frac{n^6 p^2}{\rho v_a^3}\frac{\pi^2 L^2}{2\lambda^2}\Pi_a} = \sqrt{\frac{\Pi_a}{\Pi_0}}; \qquad \Pi_0 = \frac{2\lambda^2}{\pi^2 L^2 M_2}, \quad M_2 = \frac{p^2 n^6}{\rho\, v^3}$$

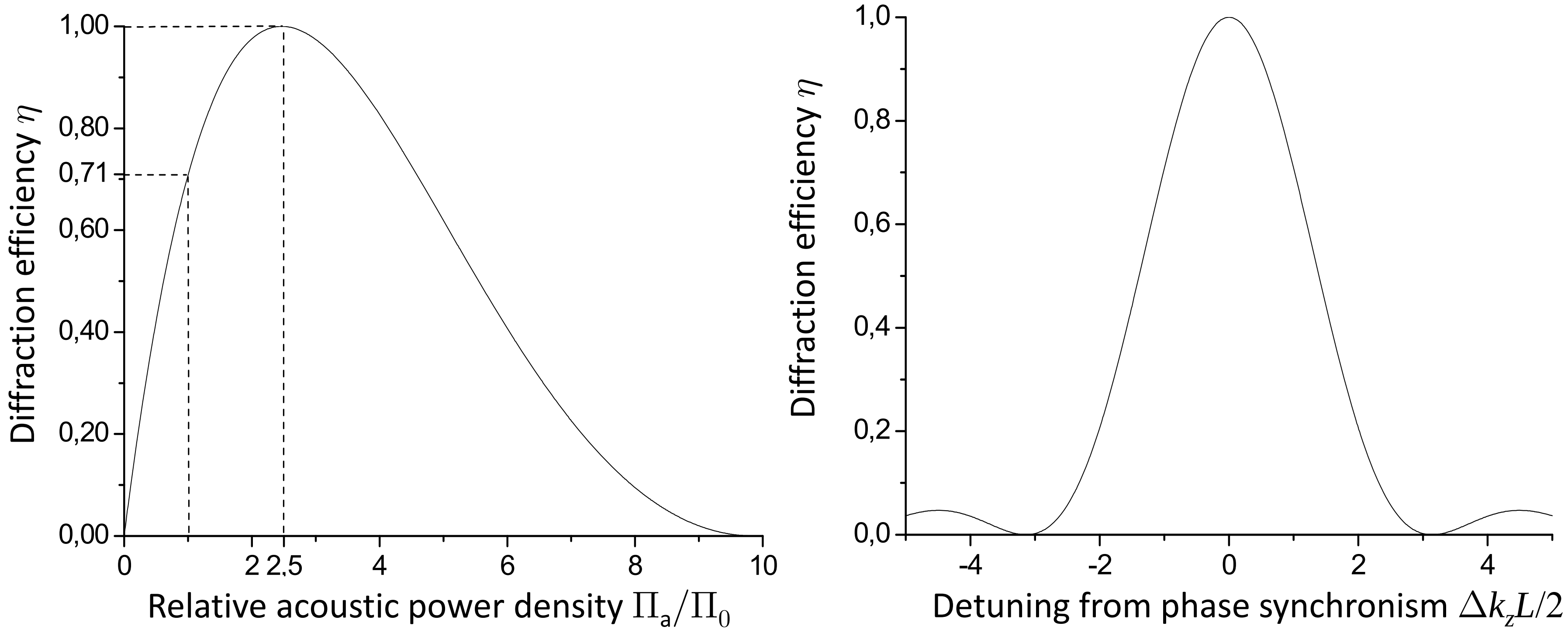
M_2 is an *acousto - optic figure of merit* of a material

At phase synchronism, $\Delta k_z L = 0$, the diffraction efficiency is $\eta = \sin^2\frac{\Delta\varphi}{2} = \sin^2\sqrt{\frac{\Pi_a}{\Pi_0}}$.

For $\eta \leq$ 0.7, the efficiency can be aproximated with $\eta \approx \frac{\Pi_a}{\Pi_0}\left(\frac{\sin\left(\Delta k_z L/2\right)}{\Delta k_z L/2}\right)^2$.

This allows to separate the effects of phase synchronism and acoustic power density.

Diffraction efficiency in the Bragg regime



For for $\Pi_a = \Pi_0$, the efficiency reaches the value of about 71% .

However, to reach the efficiency of 100% requires $\Pi_a = 2.47\Pi_0$.

Typical diffraction efficiency of acousto-optic devices is therefore usually only in the range of 80-90%, higher efficiency is quite rare.

Technical applications of acousto-optic devices

Classification according to purpose*:*

1. Deflectors of laser beam; *the diffraction angle is the function of acoustic frequency*
2. Intensity modulators of laser beam: *diffraction efficiency is ~proportional to the acoustic power*
3. Acousto-optic tunable filters: *the phase synchronism is wavelength-dependent*
4. Acousto-optic devices for processing of *electronic signals*

Classification according to interaction type

1. Devices utilizing isotropic diffraction (*in an optically isotropic medium*)
   a) with a uniphase acoustic transducer
   b) with a phased-array of acoustic transducers (flat, stepwise)
2. Devices utilizing anisotropic diffraction (*in an optically anisotropic and chiral media*)
3. Devices utilizing diffraction *on a standing acoustic wave*

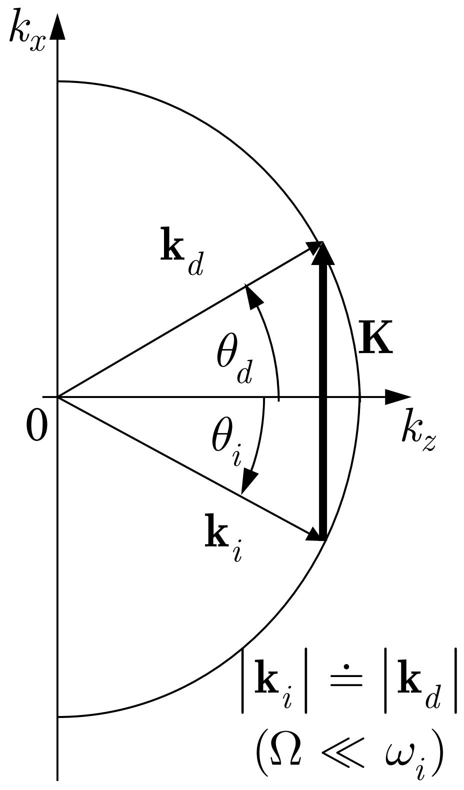
Classification according to construction

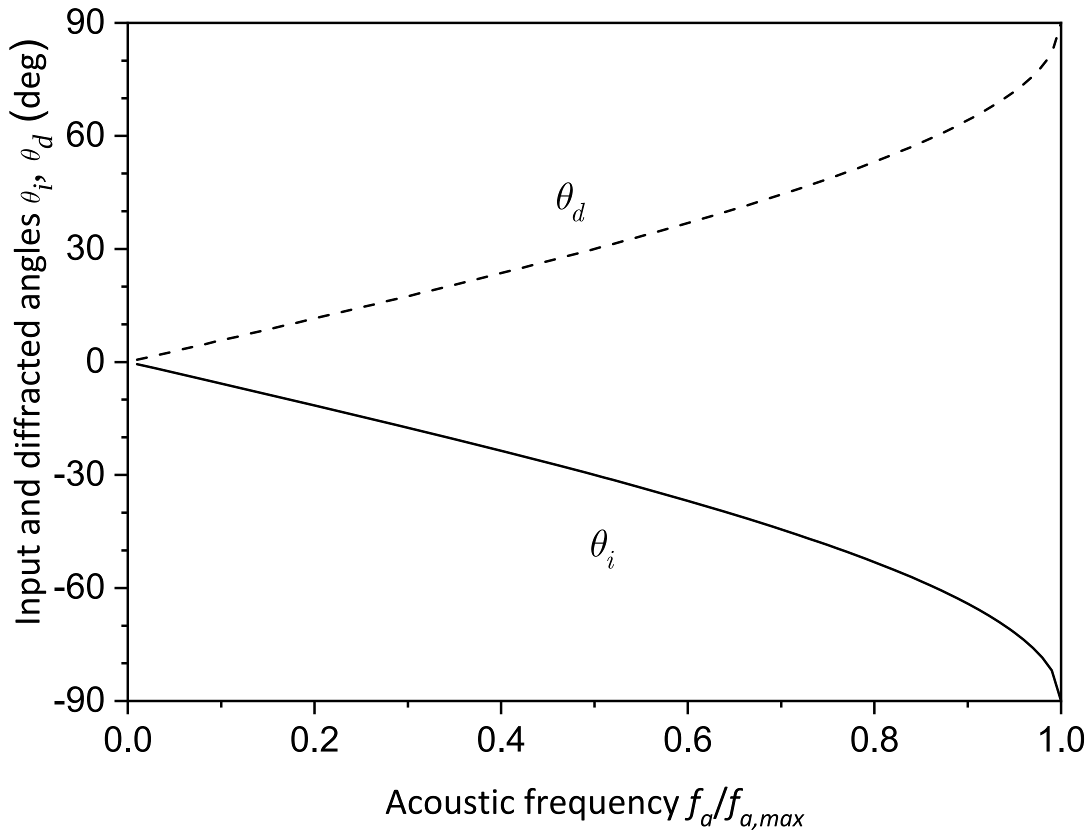
1. Bulk devices
2. Waveguide devices (*integrated-optic*)

Acousto-optic interaction in the Bragg regime in an isotropic medium

$$\theta_d = -\theta_i = \arcsin\frac{K}{2k} = \arcsin\frac{\lambda f_a}{2v_a n} = \arcsin\frac{f_a}{f_{a,\max}}, \qquad f_{a,\max} = \frac{2v_a n}{\lambda}$$

The Bragg condition (i.e., the angle of incidence) is adjusted for each frequency

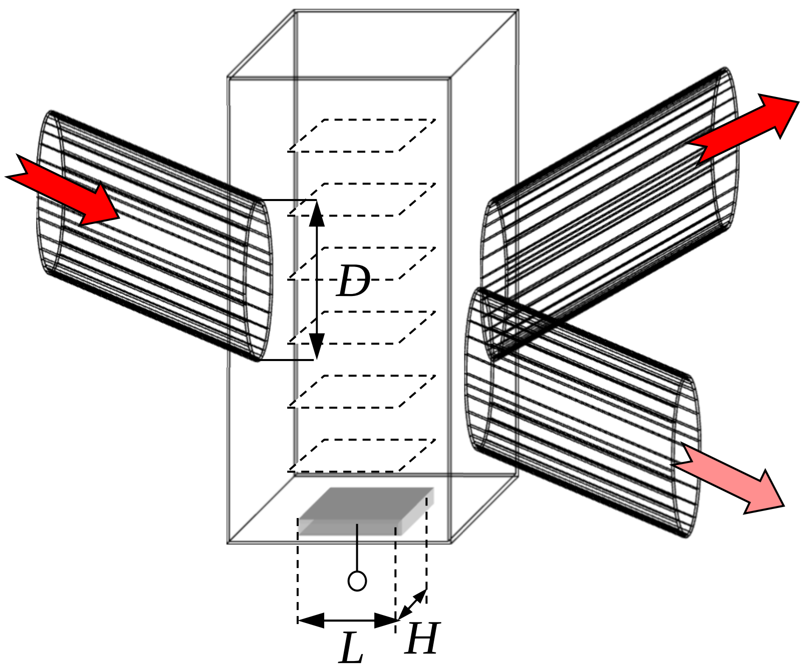




Acousto-optic deflectors of laser beams:

the angle of diffraction changes with acoustic frequency f_a

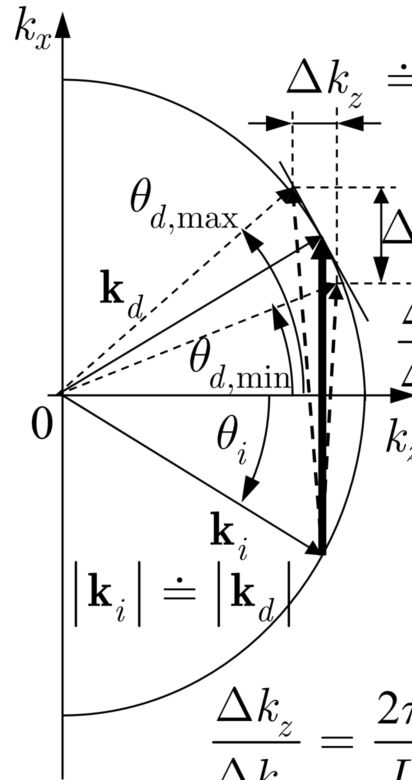
AO deflection in an isotropic medium



$$\sin\theta_d \approx \sin\theta_i + \frac{K}{k}; \qquad \theta_d \approx \theta_i + \frac{\lambda f_a}{v_a};$$

For small angles, the angle of diffracted beam is linearly dependent on acoustic frequency

Divergence angle of acoustic beam: $\Delta\theta_a \approx \dfrac{\Lambda}{L}$



$$\Delta k_z \doteq K\Delta\theta_a \doteq \frac{2\pi}{\Lambda}\frac{\Lambda}{L} = \frac{2\pi}{L}$$

$$\Delta k_x = \Delta K = \frac{2\pi\Delta f_a}{v_a}$$

$$\frac{\Delta k_z}{\Delta k_x} \approx \tan\theta_i \approx \sin\theta_i$$

$$\approx \frac{1}{2}\frac{K}{k_i} = \frac{\lambda f_a}{2nv_a};$$

Thus

$$\frac{\Delta k_z}{\Delta k_x} = \frac{2\pi}{L}\cdot\frac{v_a}{2\pi\Delta f_a} = \frac{v_a}{L\Delta f_a} = \frac{\lambda f_a}{2nv_a};$$

Total scanning angle $\Delta\theta_d \doteq \dfrac{\lambda}{v_a}\Delta f_a.$

Number of "resolvable points" (angular positions of a deflected beam):

$$N = \frac{\Delta\theta_d}{\Delta\theta_{opt}} \approx \frac{\lambda\Delta f_a}{v_a}\frac{D}{\lambda} = \frac{D}{v_a}\Delta f_a = \tau\cdot\Delta f_a, \quad \text{thus} \quad \boxed{N = \tau\cdot\Delta f_a}$$

Number of resolvable points is given by the product of the time constant and the frequency bandwidth.

Frequency bandwidth is limited by the allowable deviation from the Bragg condition:

$$\Delta f_a \le \frac{2nv_a^2}{\lambda f_a L} \quad \Longrightarrow$$ AO interaction length L should be small

But: the deflector should operate in the Bragg régime, to ensure high diffraction efficiency:

$$Q = \frac{2\pi\lambda L}{n\Lambda^2\cos\theta_i} \approx \frac{2\pi\lambda L f_a^2}{n v_a{}^2} \overset{!}{\ge} 4\pi \Longrightarrow$$ *L* must be large enough!

A compromise for the interaction length L:

$$\frac{2nv_a^2}{\lambda f_a^2} = L_{\min} \le L \le L_{\max} = \frac{2nv_a^2}{\lambda f_a \Delta f_a}.$$ Thus, $\Delta f_a \le f_a$, which is reasonable.

Acoustic power of a deflector

$$P_a = LH\Pi_0 = \frac{2\lambda^2}{\pi^2 M_2} \cdot \frac{H}{L};$$

For small acoustic power, the ratio $\frac{H}{L}$ must be small.

Hence the application of *elliptical laser beam.*

In any AO material, the acoustic wave propagates with attenuation, $\alpha \approx \Gamma f_a^2$.

AO diffraction by attenuated acoustic wave – approximate analysis

1. Reduction of the diffraction efficiency: $E_d(x) \sim \overline{S}(x) = \overline{S}_0 e^{-\alpha x}$, thus
2. Divergence of diffracted optical beam is increased due to acoustic attenuation $E_d(x) \approx E_{d,0} e^{-\alpha x}$,

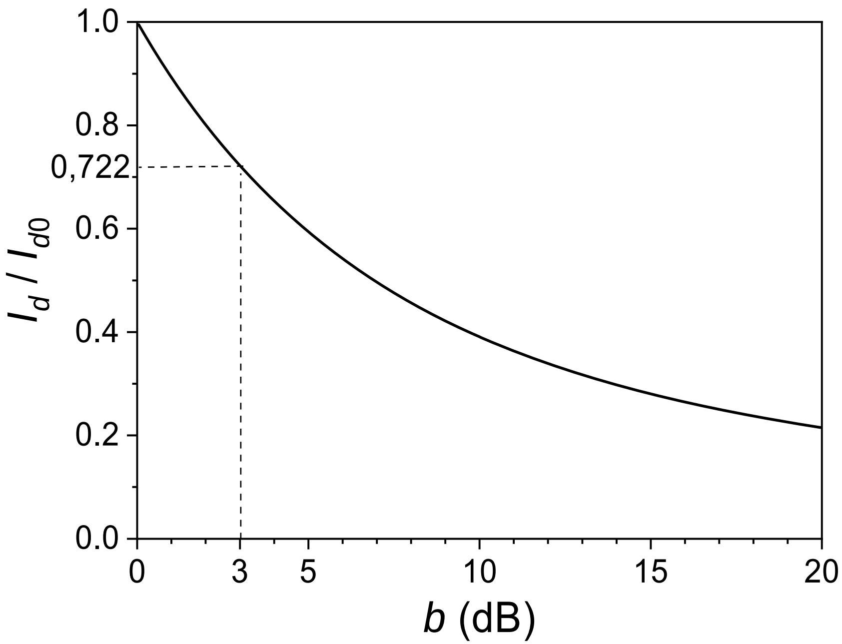
$e^{-2\alpha D} = 10^{-b/10}$; ($b = 20\alpha D/\ln 10$ [dB], $2\alpha D = b\ln 10/10$; attenuation in dB per D)

$$I_d = I_{d,0}\int_0^D e^{-2\alpha x}dx = I_{d,0}\frac{1-e^{-2\alpha D}}{2\alpha D}$$ …reduction of diffraction efficiency

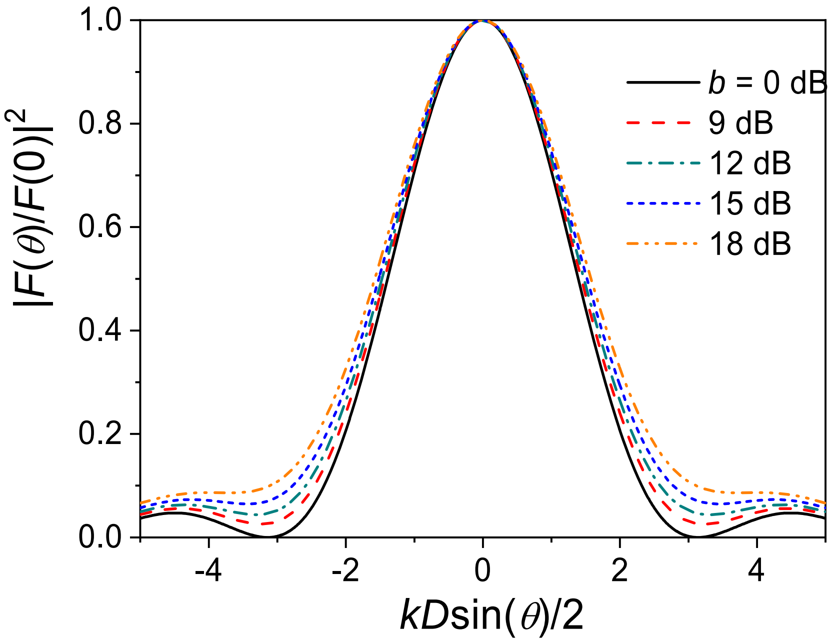
$$\left|F(\theta)\right|^2 \sim \left|\int_0^D e^{-\alpha x}e^{-ikx\sin\theta}dx\right|^2 = \frac{1+e^{-2\alpha D} - 2e^{-\alpha D}\cos^2\left(kD\sin\theta\right)}{\left(\alpha D\right)^2 + \left(kD\sin\theta\right)^2}$$ …radiation pattern

(for $\alpha = 0$ we get $\left|F(\theta)\right|^2 \sim 2D\,\mathrm{sinc}^2(kD\sin\theta)$)

Reduction of diffraction efficiency due to acoustic attentuation



Change of divergence of a diffracted beam due to acoustic attenuation

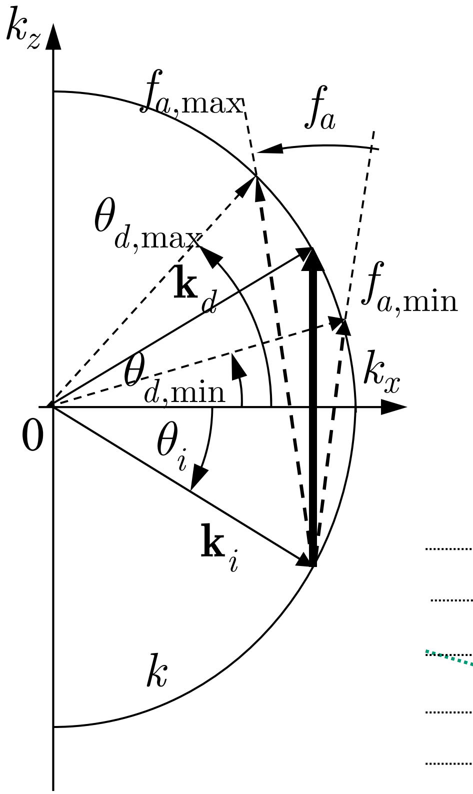


Technical solutions how to reach large number of resolvable points:

1. Strongly elliptical optical beam with large *D/H* ratio (requires complex optical with cylindrical lenses or prisms)
2. Application of a material with low acoustic velocity but small attenuation (???)
3. Ensuring generation of acoustic wave in a broad frequency range (requires high center frequency f_0)

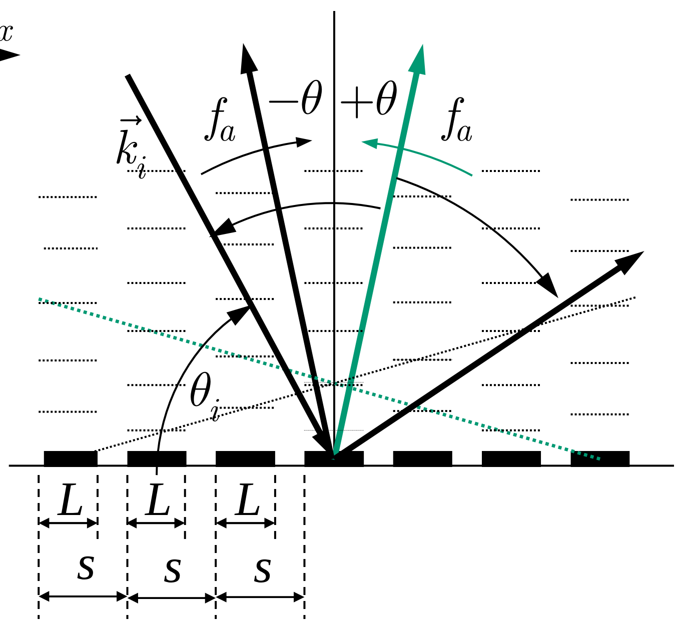
Broadening of the AO bandwidth: deflectors with acoustic beam steering

Principle: automatic "tuning" of phase synchronism while changing acoustic frequency

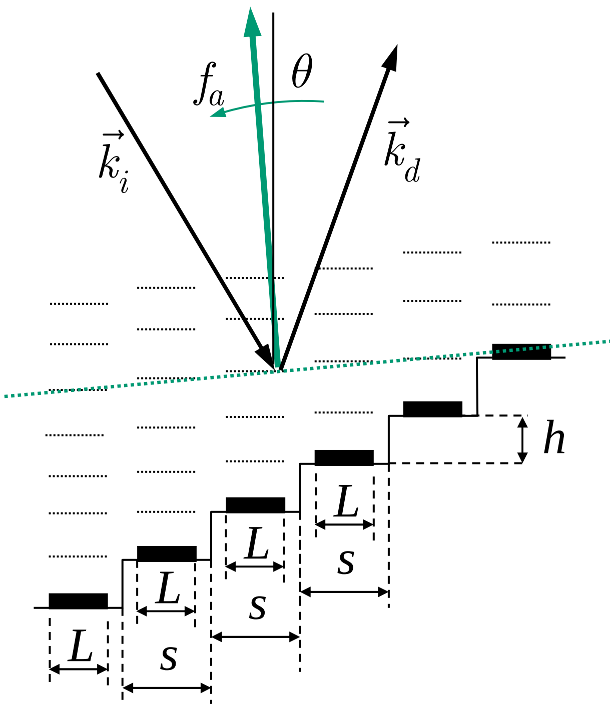


Basic approach (borrowed from microwave antennas): phased array of acoustic radiators (transducers)

1. Plane phased array



2. Staircase phased array



Difraction efficiency in the linear approximation (Gordon-Dixon method)

$$\eta \approx \frac{\pi^2}{2\lambda^2} M_2 \Pi_a L^2 \underbrace{\left(\frac{\sin\left(K\theta L/2\right)}{K\theta L/2}\right)^2}_{\text{radiation characteristics of one transducer segment}} \underbrace{\left(\frac{\sin\left[\frac{1}{2}N_m\left(K\theta s-\varphi\right)\right]}{\frac{1}{2}N_m\left(K\theta s-\varphi\right)}\right)^2}_{\text{radiation characteristics of an array of transducers}},$$

$$\theta = \theta_i + \frac{\lambda f_a}{2nv_a} = \frac{\lambda}{2nv_a}(f_a - f_{a0})$$ angle of diffraction (for small angles, sin $\theta \approx \theta$)

φ is a relative phase shift between neighbouring segments, for plane array usually $\varphi = \pi$, for staircase array $\varphi = \varphi_0 + \frac{2\pi f_a h}{v_a}$ (h is the step height)

Direction of „diffraction maxima“ $\pm\theta = \pm\frac{v_a}{2sf_a}$ for the plane array,

Staircase array has a single maximum at $\theta = \frac{h}{s} - \frac{v}{2sf_a}$ (s is the step length)

Only a half of acoustic power is utilized in a plane array, full in the staircase array.

Maximum allowabe total transducer length is now $L_{tot,\max} = N_m s \le \frac{16nv_a^2}{\lambda f_a \Delta f_a}$,

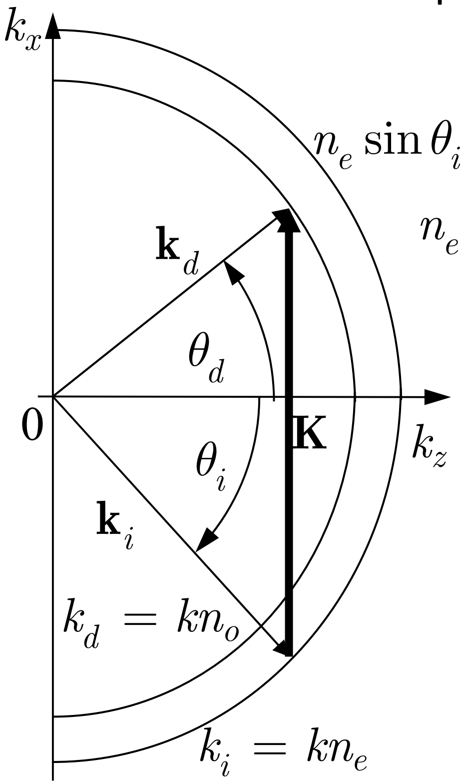
The total transducer length can be extended up to 8 times.

The acoustic power can be reduced up to 4 times with the plane array and up to 8 times with the staircase array.

Acousto-optic interaction in anisotropic media

Interaction with a transverse acoustic wave in a (positive) uniaxial medium in the plane perpendicular to the optic axis – the simple case

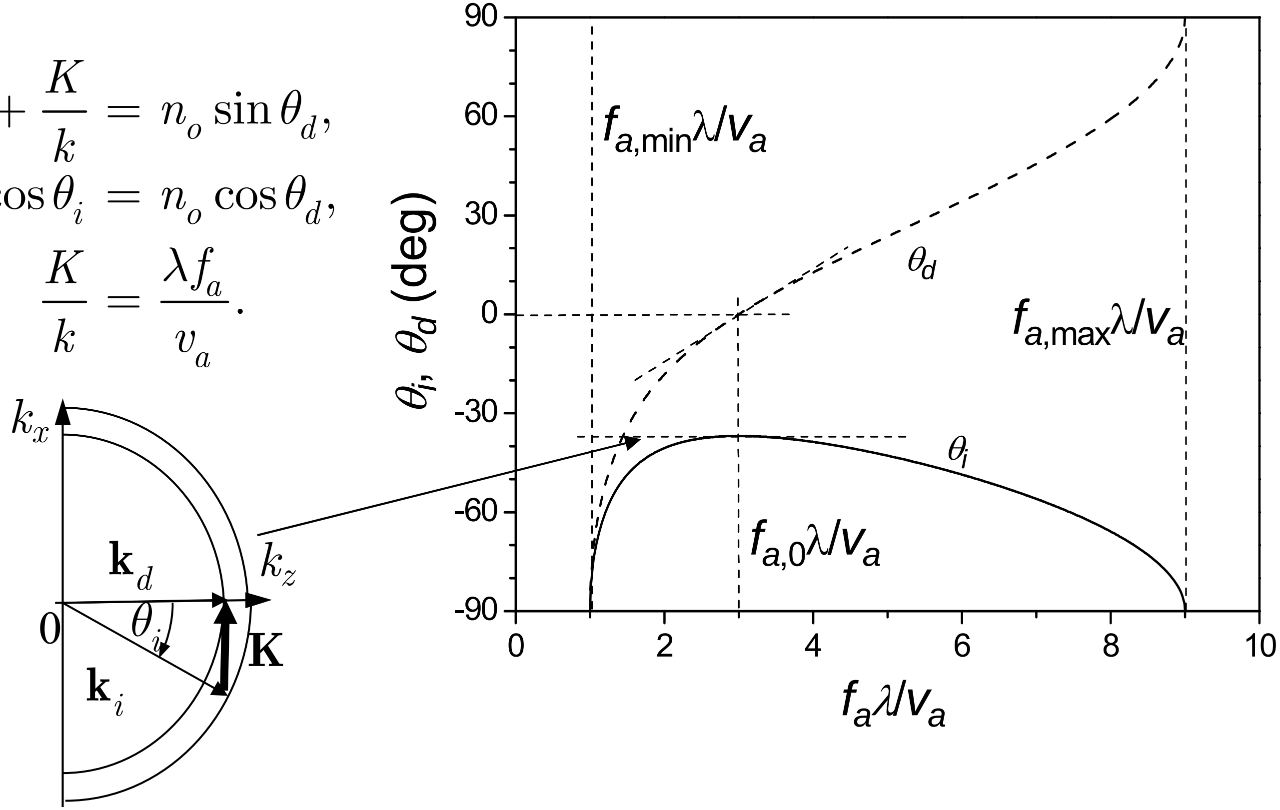
Condition of phase synchronism



$$n_e \sin\theta_i + \frac{K}{k} = n_o \sin\theta_d,$$

$$n_e \cos\theta_i = n_o \cos\theta_d,$$

$$\frac{K}{k} = \frac{\lambda f_a}{v_a}.$$



$$f_{a,\min} = \frac{v_a}{\lambda}\left|n_e - n_o\right|, \quad f_{a,\max} = \frac{v_a}{\lambda}\left(n_e + n_o\right), \quad f_{a,0} = \frac{v_a}{\lambda}\sqrt{n_e^2 - n_o^2}$$

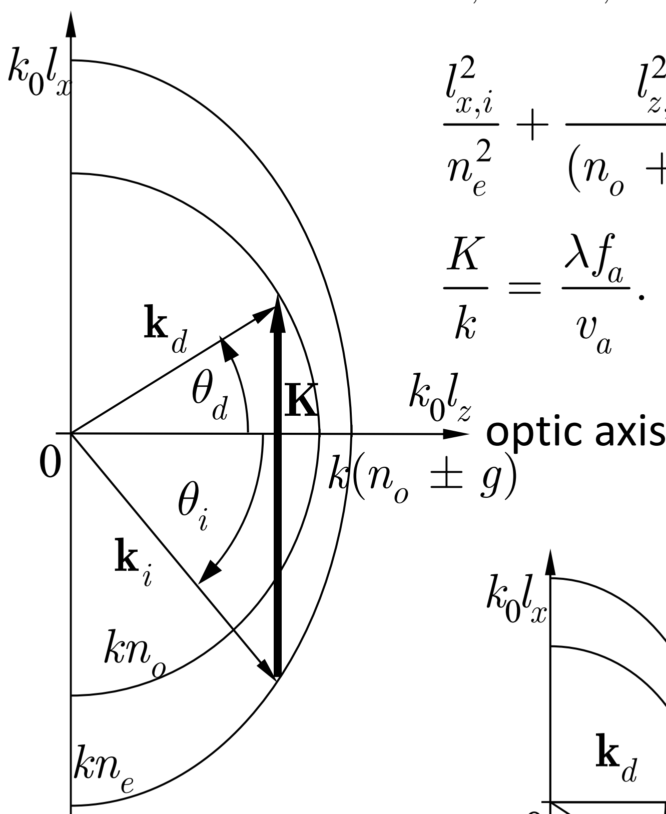
Disadvantage: Frequency $f_{a,0}$ is too high in all suitable AO materials

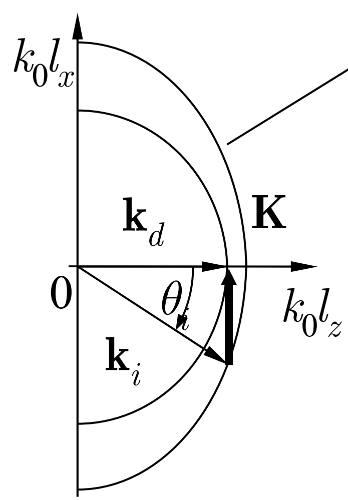
Acousto-optic interaction in a uniaxial chiral anisotropic medium

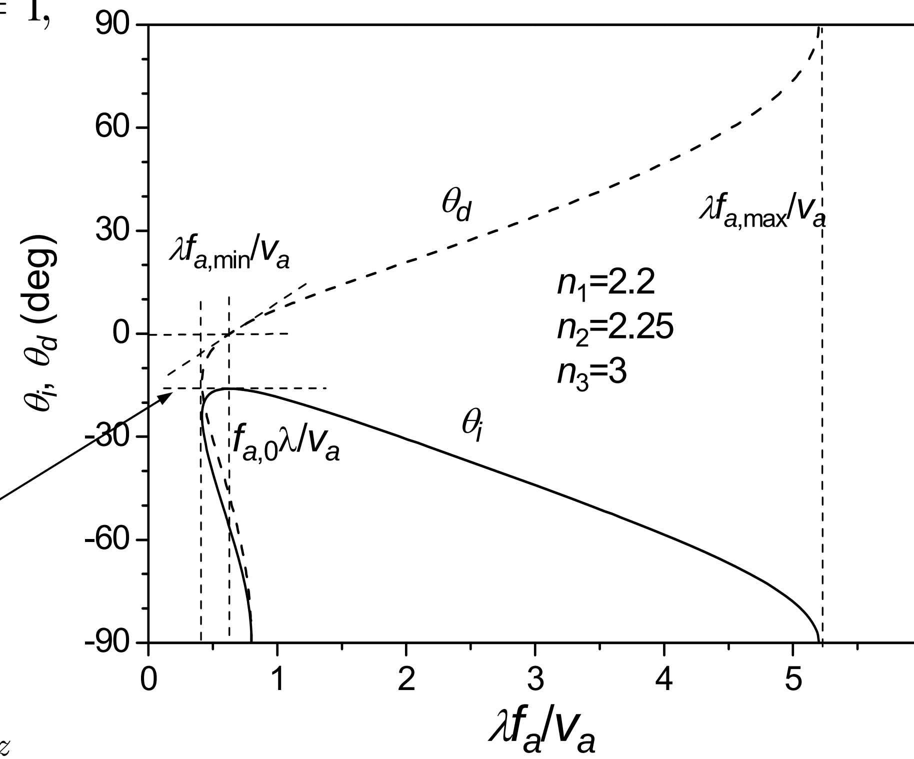
$$l_{x,d} = l_{x,i} + \frac{K}{k} \doteq (n_o + g)\sin\theta_d, \qquad \theta_i = \arctan\frac{l_{x,i}}{l_{z,i}}$$

$$\frac{l_{x,i}^2}{n_e^2} + \frac{l_{z,i}^2}{(n_o + g)^2} \doteq 1,$$

$$\frac{K}{k} = \frac{\lambda f_a}{v_a}.$$



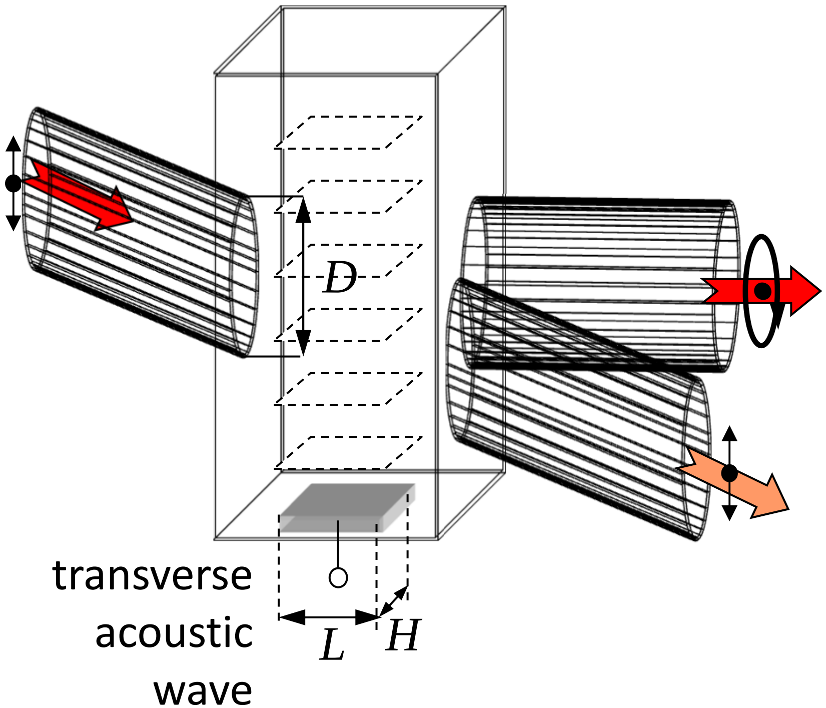




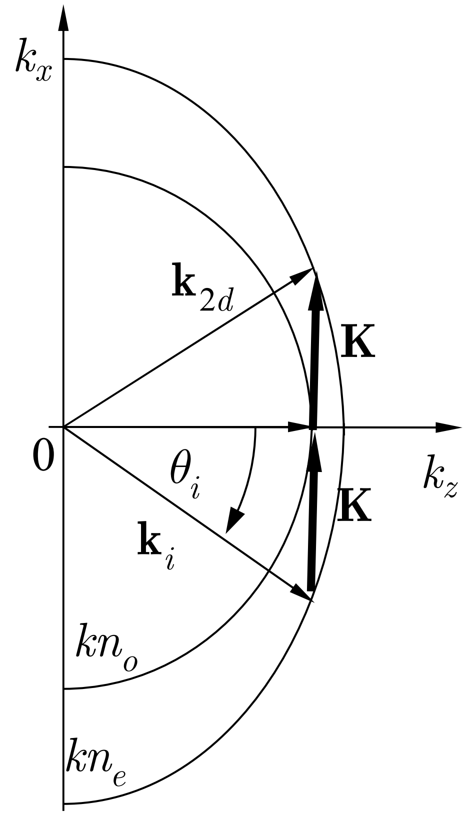
AO deflection in a chiral anisotropic medium

Advantages: reduction of angular selectivity ⇒ increase of interaction length ⇒ power reduction

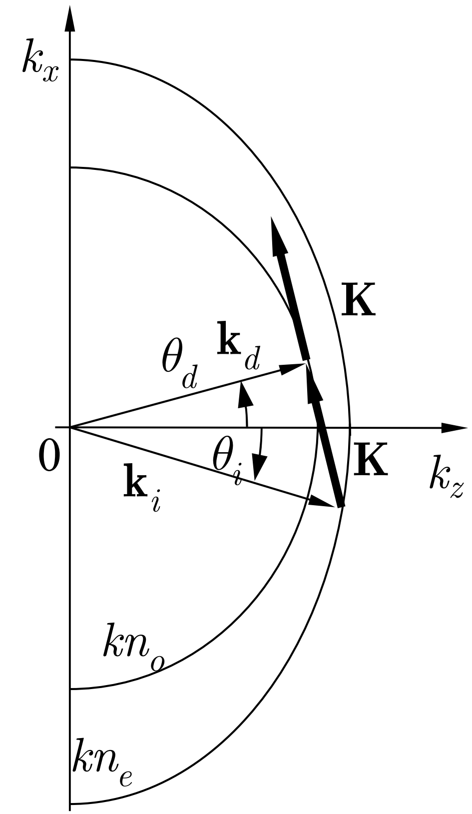
Diffraction by transverse acoustic wave ⇒ lower acoustic speed
⇒ increase of number of resolvable points, diffracted wave differs in polarization



$$L_{\max} \leq \frac{16 n v_a^2}{\lambda f_a \Delta f_a}$$



Degenerate AO
interaction of the 2nd order

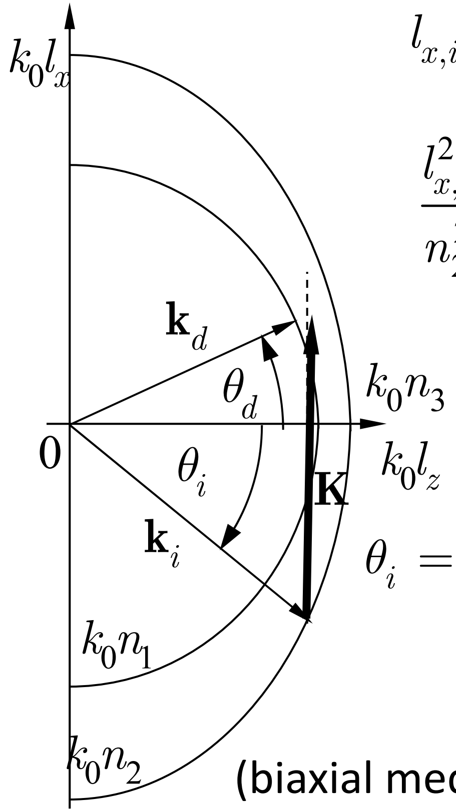


Suppression of the diffraction
into the 2nd order

Frequency dependence of the diffraction efficiency in the “optimum” configuration of anisotropic AO diffraction

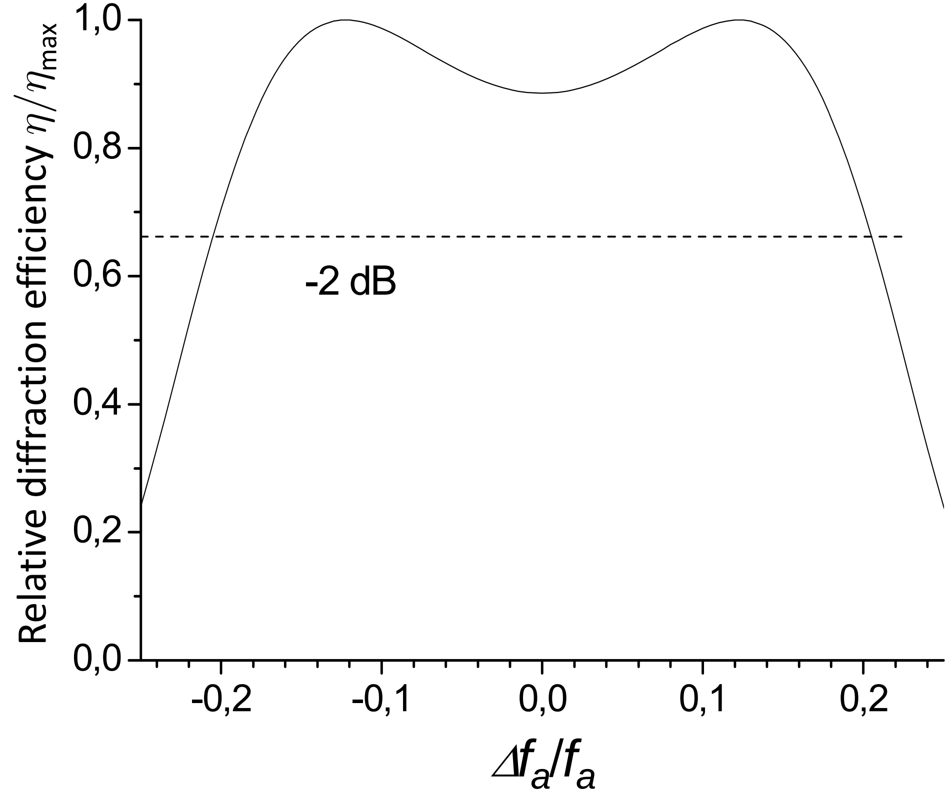
$$l_{x,i} + \frac{\lambda f_a}{v_a} = n_1 \sin\theta_d; \quad \Delta k_z = k_0 \left| \sqrt{n_1^2 - \left(\frac{\lambda f_a}{v_a}\right)^2} - l_{x,i} \right|,$$

$$\frac{l_{x,i}^2}{n_2^2} + \frac{l_{z,i}^2}{n_3^2} = 1$$



(biaxial medium)

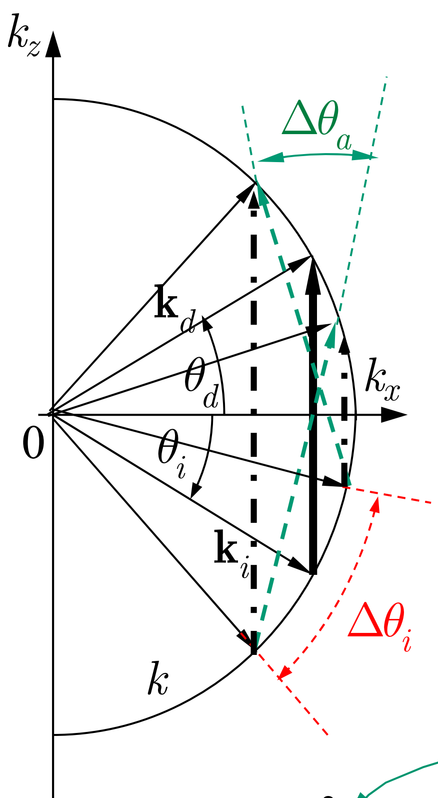
$$\theta_i = \arctan\frac{l_{x,i}}{l_{z,i}}$$



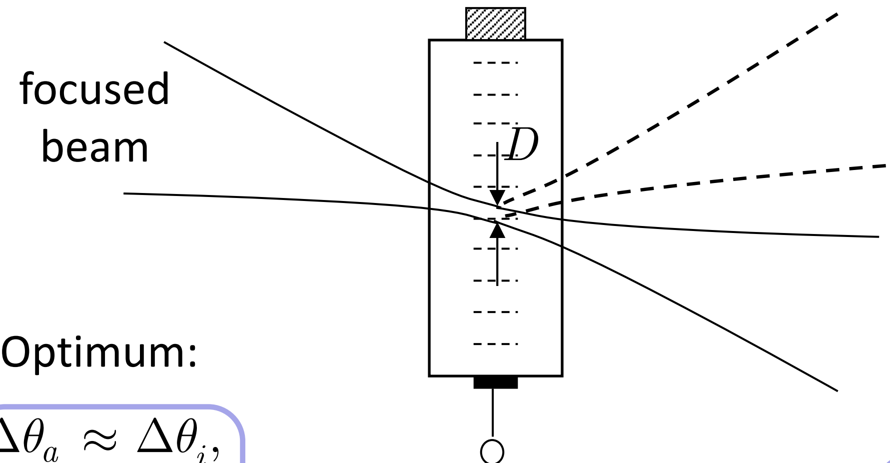
The shape of the curve can be modified by adjusting the angle of incidence of the input optical beam

Acousto-optic modulation

Acousto-optic modulation is, in fact, complementary to acousto-optic deflection



Light intensity modulation of an optical beam takes place only *if the waves diffracted by different frequencies overlap in space*



Optimum:

$$\Delta\theta_a \approx \Delta\theta_i,$$

$$\frac{\Lambda_a}{L} = \frac{v_a}{f_a L} \approx \frac{\lambda}{nD}, \quad L \approx \frac{n v_a D}{\lambda f_a}. \quad \tau \approx \frac{1}{\Delta f_a} \approx \frac{D}{v_a}, \quad D \approx \frac{v_a}{\Delta f_a}.$$

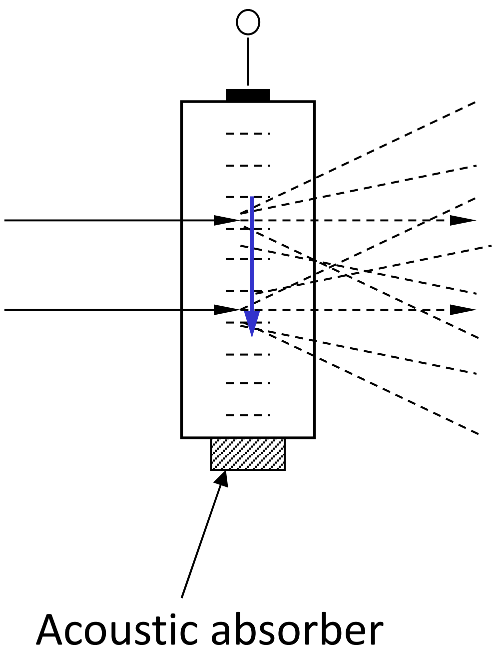
$$L \approx \frac{n v_a^2}{\lambda f_a \Delta f_a} > L_{\min} = \frac{2 n v_a^2}{\lambda f_a^2}$$ … condition for Bragg regime, so that $\Delta f_a < f_a / 2.$

For $\Delta f_a = 50\ \mathrm{MHz}$, $\tau \approx 20\ \mathrm{ns}$ we get $D \approx \frac{v_a}{\Delta f_a} \approx \frac{3\times 10^3}{50\times 10^6} = 60\ \mu\mathrm{m}.$

Acousto-optic modulators for Q-switching and mode-locking of lasers

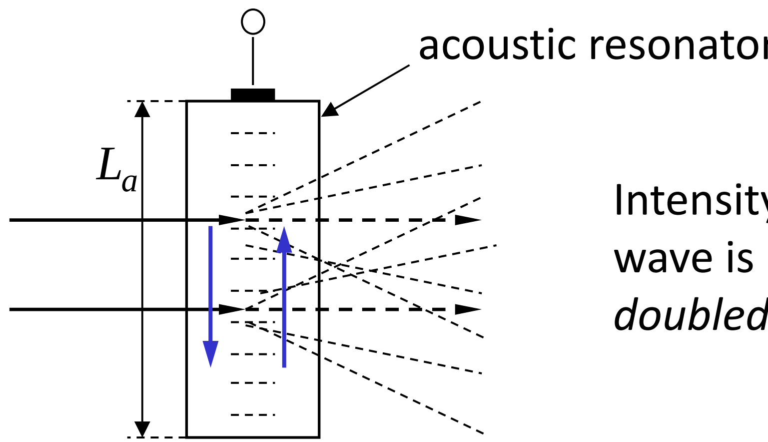
Both Raman-Nath and Bragg regimes are applicable; suppression of the 0th order is sufficient.

Q-switching: diffraction by *running acoustic wave* (with a *moderate efficiency*) is required.



Mode locking: diffraction by *standing acoustic wave*

$$f_a = \frac{qv_a}{2L_a};\ \Delta f_{opt} = \frac{c}{2nl_{laser}};\ f_a = \frac{\Delta f_{opt}}{2}$$



Intensity of the transmitted wave is modulated by a *doubled acoustic frequency*

$$\Delta\bar{\varepsilon}(x,t) = \Delta\bar{\varepsilon}\left[\sin\left(Kx-\Omega t\right)+\sin\left(-Kx-\Omega t\right)\right] =$$

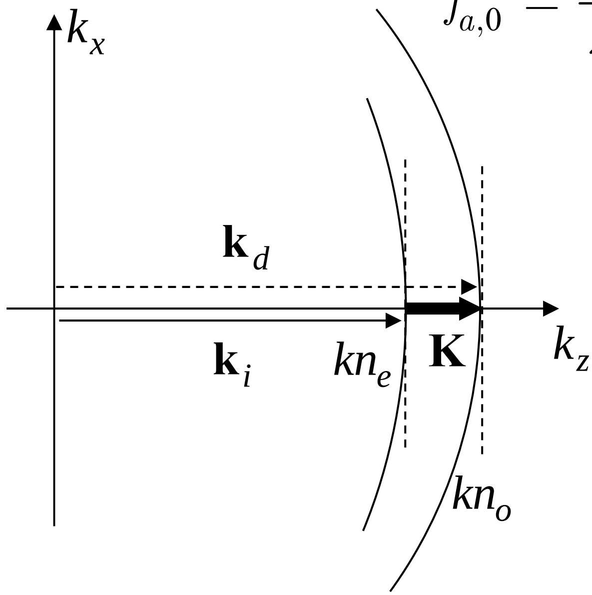
$$= \Delta\bar{\varepsilon}\left[\sin\left(Kx-\Omega t\right)-\sin\left(Kx+\Omega t\right)\right] = -2\Delta\bar{\varepsilon}\cos(Kx)\sin\Omega t.$$

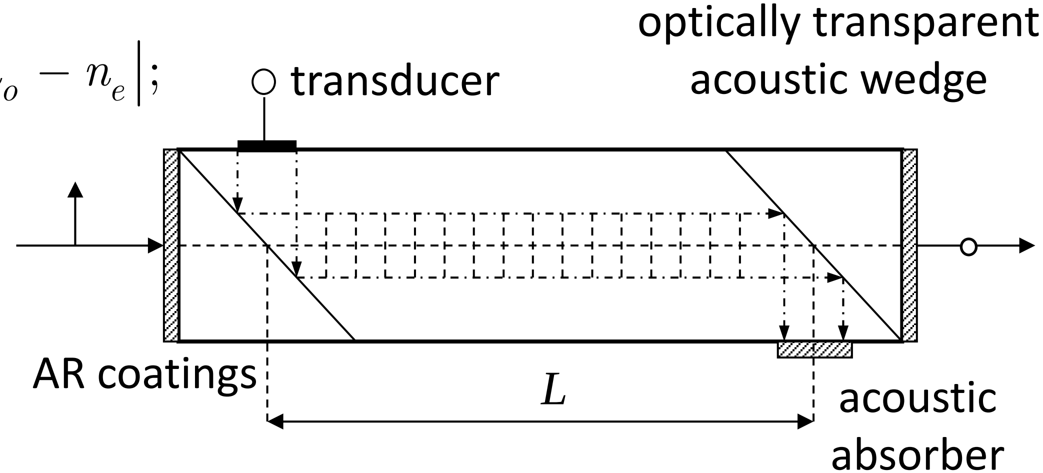
$$I_0 = I_0\left[\left(\Delta\bar{\varepsilon}\right)^2\right] = f\left[\sin^2\left(\Omega t\right)\right] = f\left[(1-\cos 2\Omega t)/2\right] = g\left(2\Omega t\right).$$

Acousto-optic tunable filters

Basic configuration: **collinear AO interaction**

$$f_{a,0} = \frac{v_a}{\lambda}\left|n_o - n_e\right|;$$





$$\Delta k_z L = \Delta\left(\frac{2\pi}{\lambda}L\left|n_0 - n_e\right|\right) \approx \frac{2\pi}{\lambda^2}L\left|n_0 - n_e\right|\Delta\lambda$$

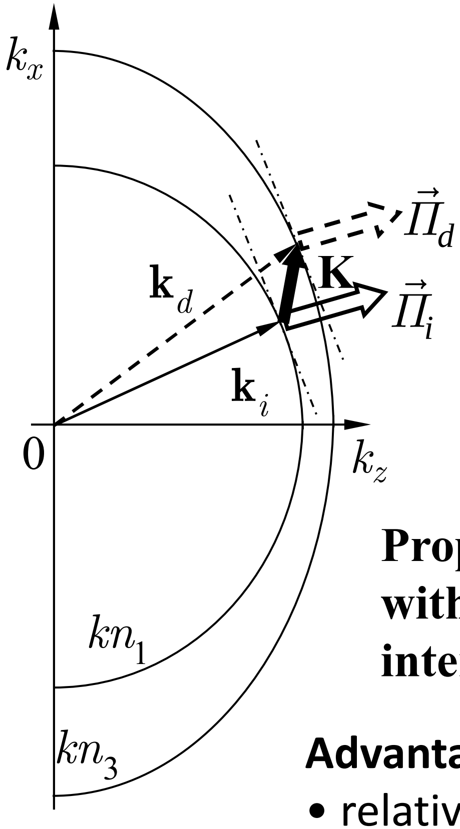
$$\Delta k_z L \,/\, 2 \le \pi \,/\, 2 \;\Rightarrow\; \Delta\lambda \approx \frac{\lambda_0^2}{2\left|n_0 - n_e\right|L}$$

$$\lambda = \frac{v_a}{f_a}\left|n_o(\lambda) - n_e(\lambda)\right| \approx \frac{v_a}{f_a}\left|n_o(\lambda_0) - n_e(\lambda_0)\right|$$

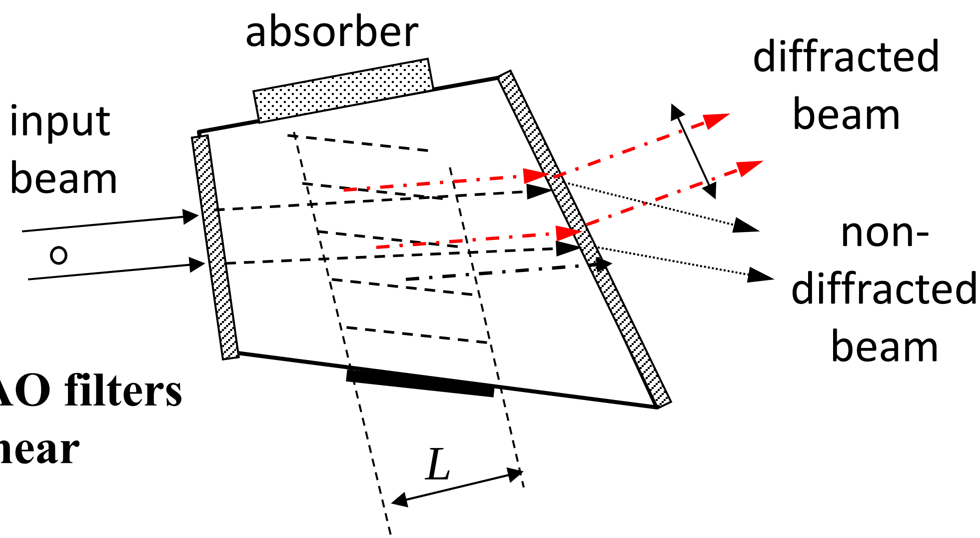
Advantages of the collinear interaction
- narrow optical bandwidth
- relatively large optical aperture

Disadvantages:
- high average acoustic frequency
- complicated design

Acousto-optic filters with non-collinear interaction



General design:



Properties of AO filters with non-collinear interaction

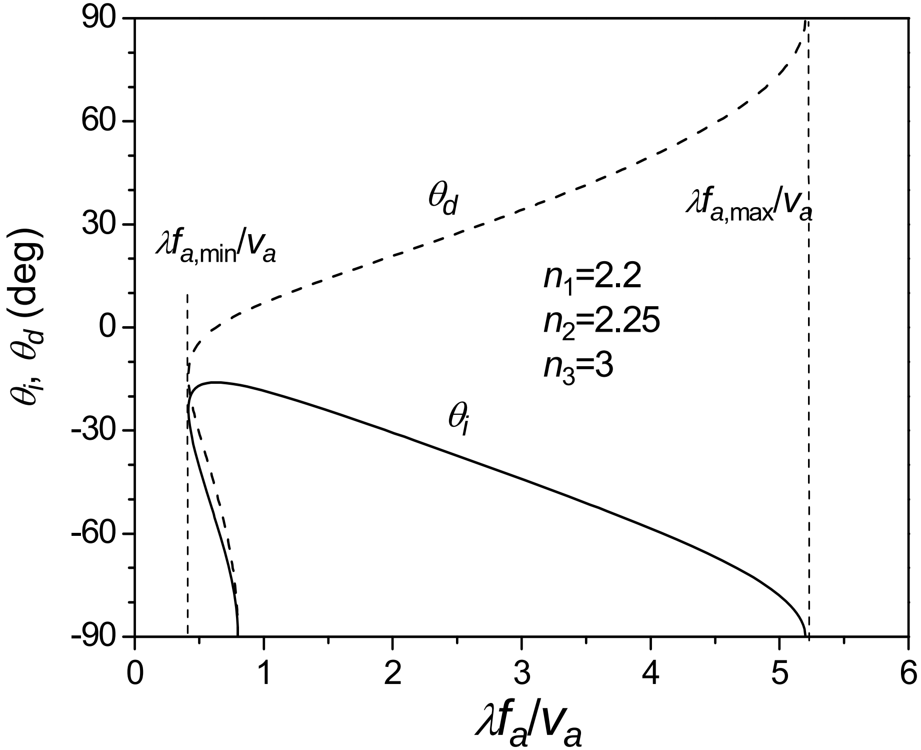
Advantages:

- relatively large angular aperture
- directions of propagation of incident and diffracted beams are identical so that the interaction is efficient
- “filtered” (diffracted) beam has different polarization
- *high flexibility* of design by configuration change

Disadvantages:

- lower spectral selectivity,
- bandwidth inreases with the *square of the optical wavelength*

Optimum configuration of a non-collinear AO tuneable filter with minimum frequency



Acoustic frequency of the filter is close to the minimum frequency for the non-collinear AO interaction

Typical parameters of AO filters based on TeO_2 crystals:

$$\Delta\lambda \approx 10 \div 100 \text{ nm}$$

$$f_{a,0} \approx 40 \div 200 \text{ MHz}$$

$$NA \approx 10' \div 20^\circ$$

$$P_a \leq 0{,}2 \div 2 \text{ W}$$

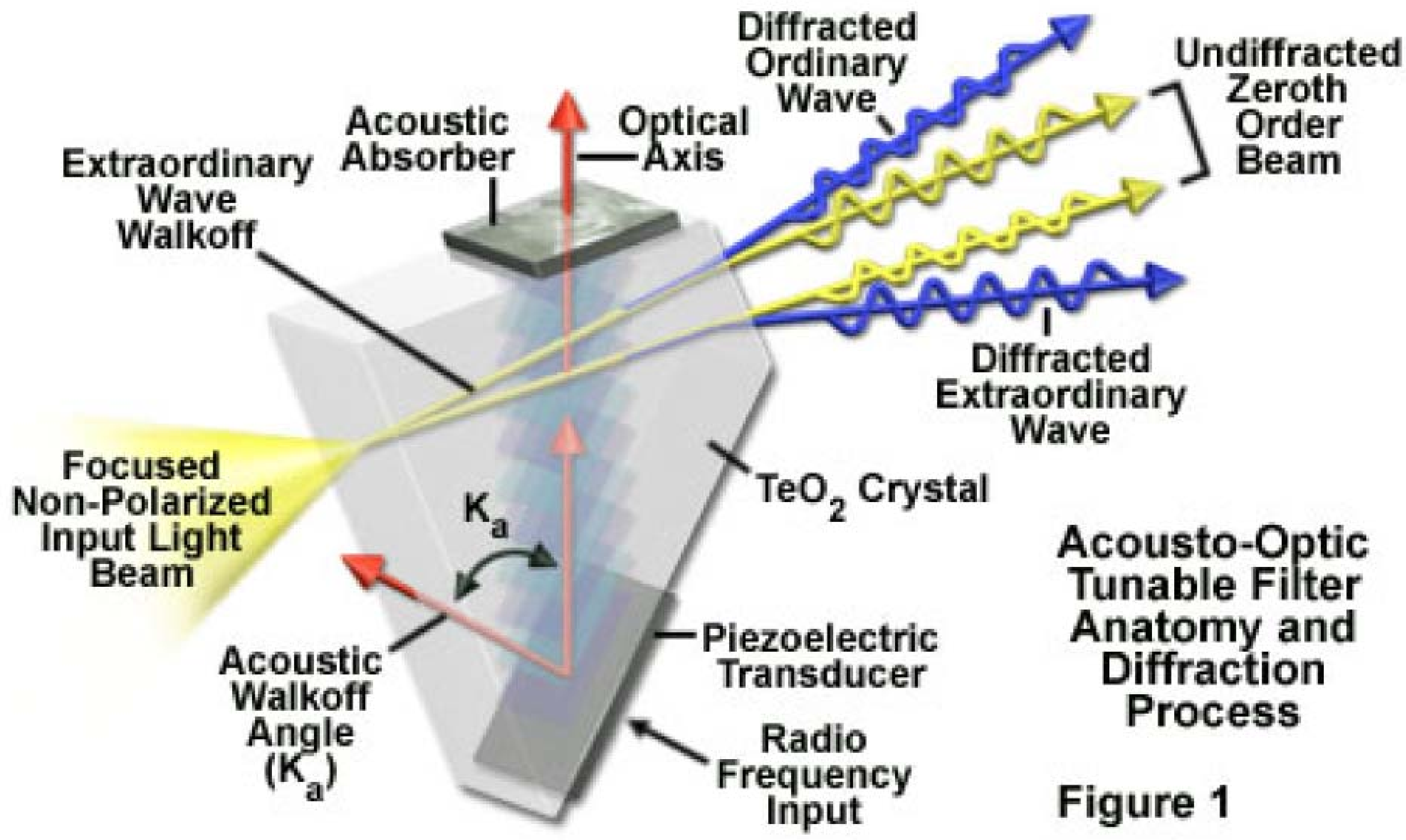
$$\lambda \approx 0{,}4 \div 10 \text{ μm}$$

Selected AO materials

Material	optical window (µm)	n (n_o, n_e)	$M_2 \times 10^{15}$	v_a (km/s)	Z_a (kg/m^{2}s)	Type of acoustic wave
Fused quartz	0,2 4,5	1,457	1,56 ‖ 0,47 ⊥	5,96 3,76	13,12	L
SF59 glass	0,46 2,5	1,95	19,1	3.26	20,5	L
$LiNbO_3$	0,5 4,5	2,202 2,286	7	6,57	30,6	L
$PbMoO_4$	0,4 5,5	2,262 2,386	36,3‖ 36,1 ⊥	3,63	25,22	L
TeO_2 para-tellurite	0,35 5	2,26 2,412 opt. akt.	34.5 ⊥ 25,6‖ **1200**	4.2 **0.616**	25,2 3,7	L S
Hg_2Cl_2 calomel	0,4 30	**1,97** **2,65**	506 640	1,62 **0.34**	11,6 2,4	L S
GaP	0,6 10	3,31	44,6	6,32	26,1	L

Acousto-optic tuneable filters for confocal microscopy (OLYMPUS)

Non-collinear filter based on TeO_2



Acousto-optic tuneable filters for confocal microscopy (OLYMPUS)

Collinear filter based on the SiO_2 single-crystal

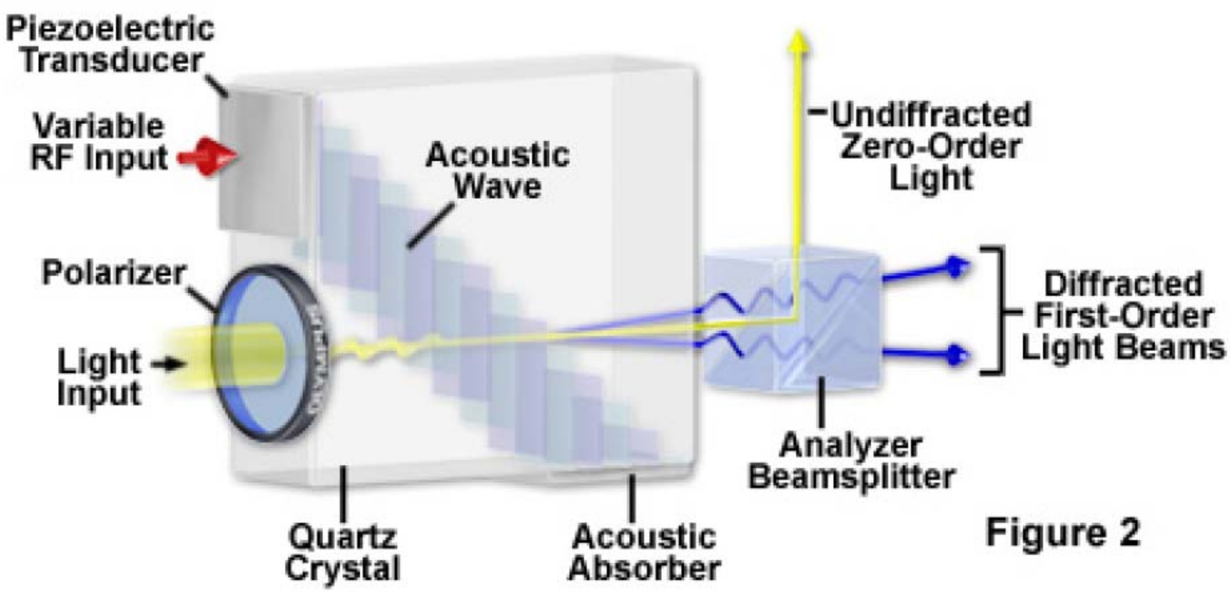


Figure 2

AO tuneable filter based on KDP for UV-VIS region

Parameters of KDP Crystal

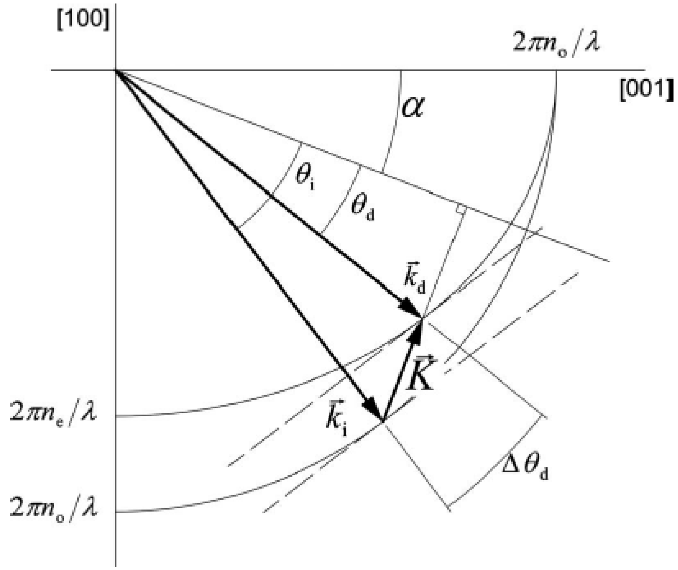
Parameter						
At Wavelength		633 nm	480 nm	350 nm	220 nm	200 nm
Index of Refraction	n_o	1.507	1.515	1.532	1.596	1.622
	n_e	1.467	1.470	1.487	1.543	1.562

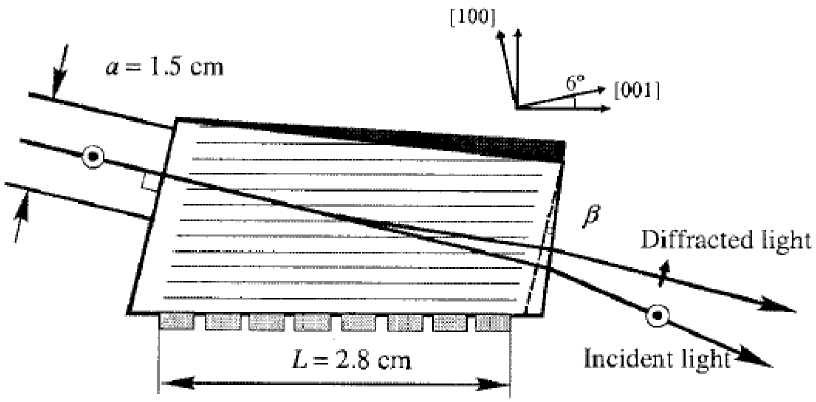
Density ρ = 2.34 g/cm^3

Effective photoelastic coefficient at 12° relative to *Z* axis in *XZ* plane, p_{eff} = 0.067

Acoustic phase velocity at 6° relative to *X* axis in *XZ* plane, v = 1.66×10^5 cm/s

AO figure of merit M_2 = 4.6×10^{18} s^3/g





Spectral range: 220–650 nm

Spectral passband at 350 nm: 2 nm; at 633 nm: 67 nm

Rf range, 60–164 MHz

Linear aperture: 1.5×1.5 cm^2

Angular aperture: 1.2°

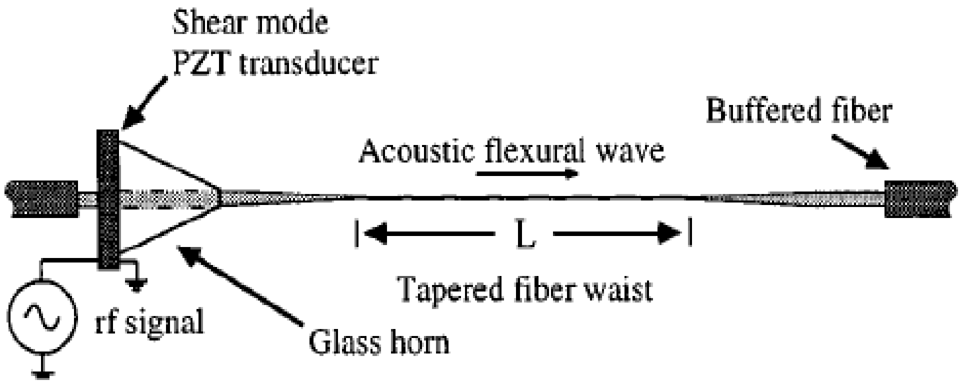
Applied power: 2.0 W

Transmission coefficient: 60%

N. Gupta, V. Voloshinov, APPLIED OPTICS, Vol. 43, No. 13, pp. 2752-2759, 2004

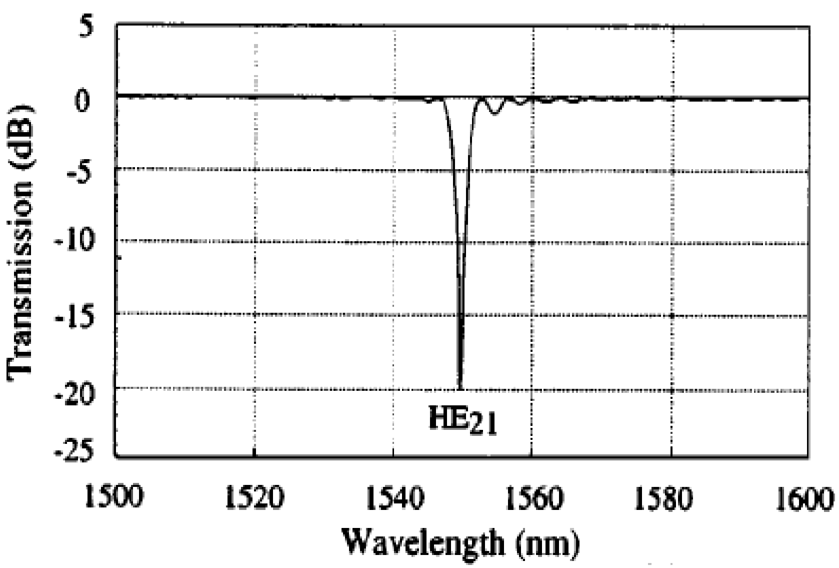
Acousto-optic „notch“filter in an optical fibre

(Photonics Technol. Lett., Sept. 2000)

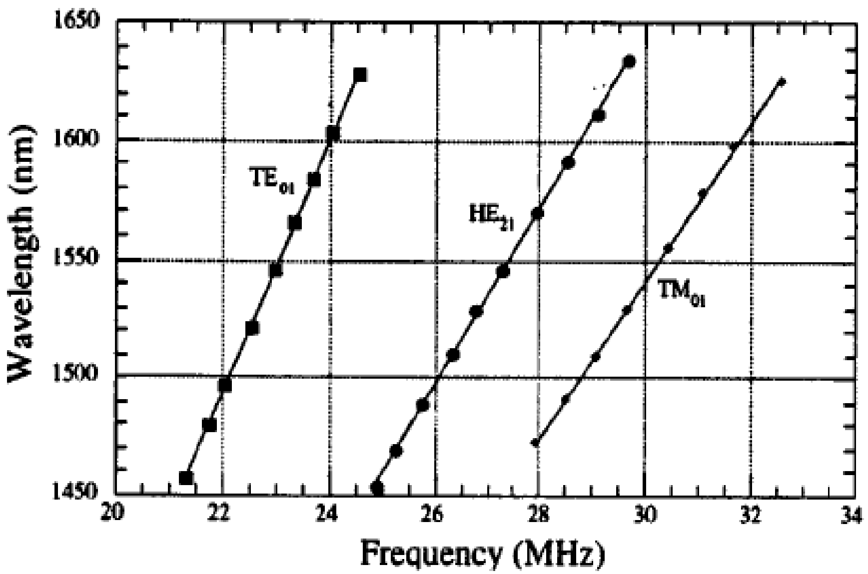


Transverse acoustic wave creates periodic microbends, which cause coupling from the core into cladding modes

Spectral transfer function

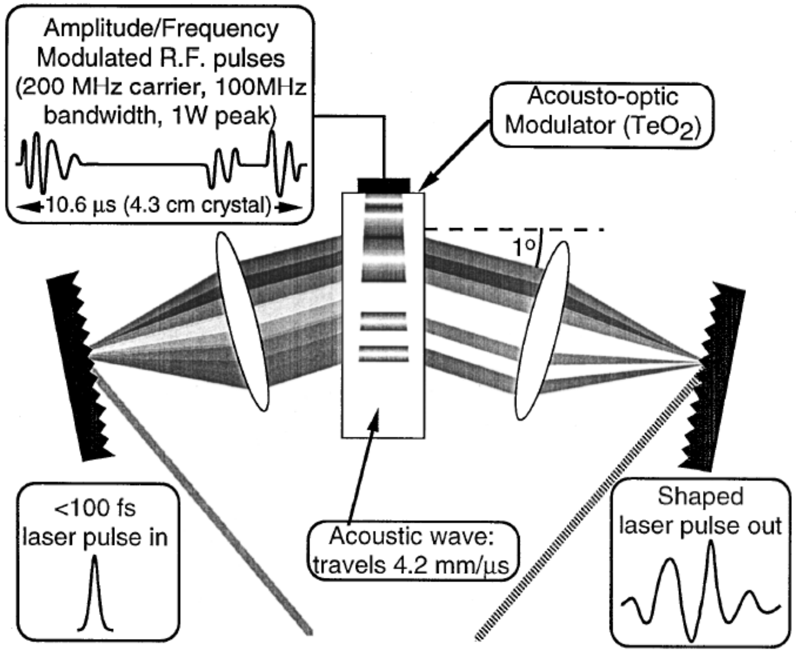


Tuning curve

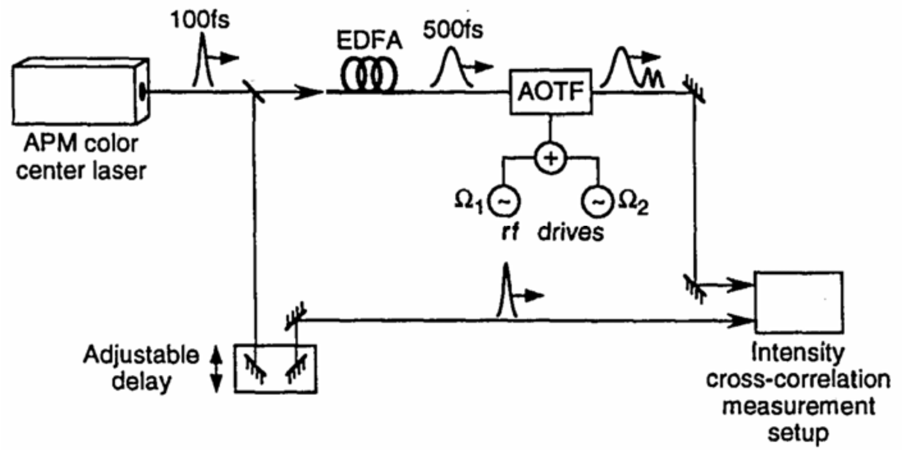


AO control of dispersion of ultrashort optical pulses

acousto-optic programmable dispersive filter (AOPDF)

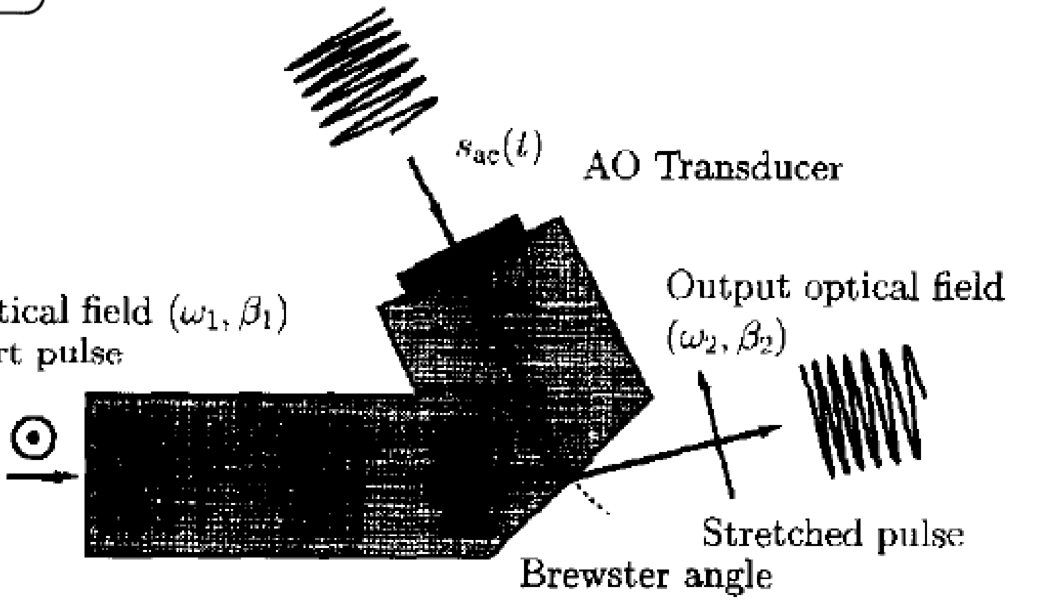


M. A. Dugan, J. X. Tull, W. S. Warren
JOSA B **14**,2348-2358, 1997



M. E. Fermann, V. da Silva, D. A. Smith,
Y. Silberberg, A. M. Weiner
Optics Letters 18, 1505-1507, 1993

P. Tournois: Optics Communications
140, 245-249, 1997



Acousto-optic dispersion control of ultrashort optical pulses ("Dazzler" fundamentals)

M. E. Fermann, V. da Silva, D. A. Smith, Y. Silberberg, A. M. Weiner: Shaping of ultrashort optical pulses by using an integrated acousto-optic tunable filter. Optics Letters **18**, 1505-1507, 1993

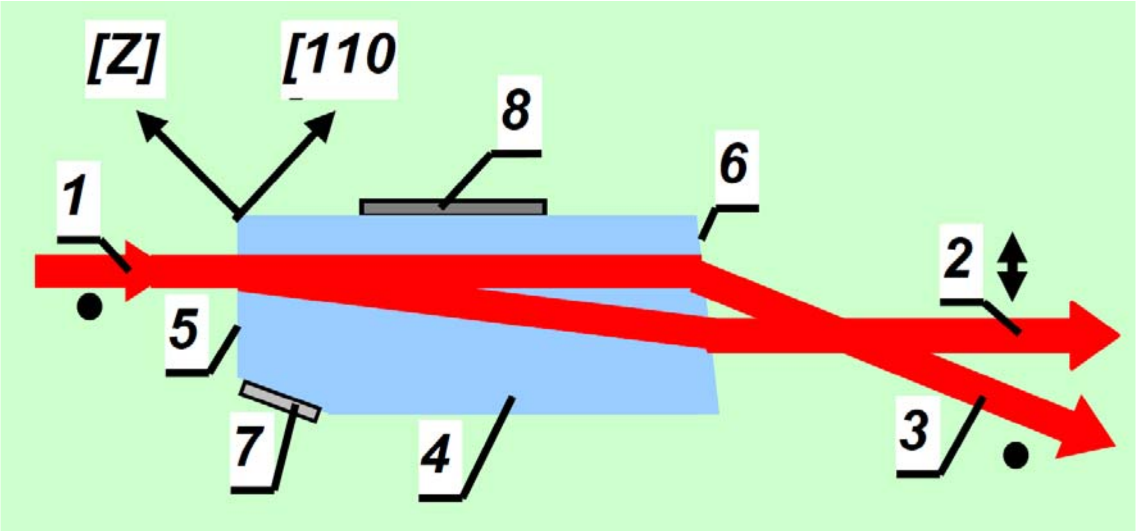
M. A. Dugan, J. X. Tull, W. S. Warren: High-resolution acousto-optic shaping of unamplified and amplified femtosecond laser pulses. JOSA B **14**,2348-2358, 1997

P. Tournois: Acousto-optic programmable dispersive filter for adaptive compensation of group delay time dispersion in laser systems. Optics Communications **140**, 245-249, 1997

F. Verluise, V. Laude, J.-P. Huignard, P. Tournois: Arbitrary dispersion control of ultrashort optical pulses with acoustic waves. JOSA B, **17,** 138-145, 2000.

F. Verluise, V. Laude, Z. Cheng, Ch. Spielmann, P. Tournois: Amplitude and phase control of ultrashort pulses by use of an acousto-optic programmable dispersive filter: pulse compression and shaping. Optics Letters **25**, 8, 575-577, 2000.

V. Ya. Molchanov, S. I. Chizhikov, O. Yu. Makarov: Interaction between femtosecond radiation and sound in a light dispersive delay lines using effect of strong elastic anisotropy. J. Phys.: Conf. Ser. **278,** 012016, 2011.



Optical scheme of light dispersive delay line:

1. incoming beam;
2. diffracted beam;
3. non-diffracted beam;
4. TeO2 crystal;
5. input optical facet;
6. output facet;
7. transducer;
8. acoustic absorber.

Mostly used physical principle: (quasi)collinead acousto-optic interaction

Approximation used for an elementary description:

1. sufficiently strong acoustic wave
2. “non-depleted” incident beam approximation
3. (“negligibly” small acoustic velocity)

Optical radiation:

$$\mathbf{E}(z,t) = \mathbf{e}_i \int_0^\infty a_i(z,\omega) e^{-i\omega t} d\omega + \mathbf{e}_d \int_0^\infty a_d(z,\omega) e^{-i\omega t} d\omega, \quad \mathbf{e}_i \cdot \mathbf{e}_d = 0.$$

a_i complex amplitude of incident wave,
a_d complex amplitude of diffracted wave, (polarized orthogonallly)

Acoustic wave (excited by an electric signal of a general form) modulates the permittivity tensor

$$\Delta\boldsymbol{\varepsilon} \approx \Delta\boldsymbol{\varepsilon}(z) e^{iK_a(z)z}$$

Optical transit time through a device is much shorter than the period of an acoustic wave, so that we can neglect the acoustic wave motion.

Coupled equations for complex amplitudes of optical waves:

$$\frac{da_i(z,\omega)}{dz} \approx ik_i(\omega)a_i\left(z,\omega\right) + \cancel{i\kappa(z,\omega)a_d\left(z,\omega\right)}$$
$$\frac{da_d(z,\omega)}{dz} \approx ik_d(\omega)a_d\left(z,\omega\right) + i\kappa(z,\omega)a_i\left(z,\omega\right).$$

$$\kappa\left(z,\omega\right) \approx \frac{k_0}{4}\mathbf{e}_i \cdot \Delta\varepsilon(z)\cdot\mathbf{e}_d\, e^{iK_a(z)z} = \frac{\omega}{4c}\Delta\varepsilon_{id}e^{iK_a(z)z}$$

Neglecting the backward effect of the diffracted wave on the incident wave, we get

$$a_i(z,\omega) \approx a_i(0,\omega)\exp\left[ik_i(\omega)z\right];$$

Let us write the diffracted wave in the following form:

$$a_d(z,\omega) = A_d(z,\omega)\exp[ik_d(\omega)z];$$
$$\frac{da_d(z,\omega)}{dz} = \underbrace{ik_d(\omega)a_d(z,\omega)} + e^{ik_d(\omega)z}\frac{dA_d(z,\omega)}{dz} \approx \underbrace{ik_d(\omega)a_d(z,\omega)} + i\kappa(z,\omega)a_i(0,\omega)e^{ik_i(\omega)z}$$

Then $\dfrac{dA_d(z,\omega)}{dz} \approx i\kappa(z,\omega)e^{i\left[k_i(\omega)-k_d(\omega)\right]z}a_i(0,\omega)$ or

$$A_d\left(L,\omega\right) \approx i\int_0^L \kappa(z,\omega)e^{i\left[k_i(\omega)-k_d(\omega)\right]z}\,dz a_i(0,\omega);\quad a_d(L,\omega) = A_d(L,\omega)e^{ik_d(\omega)L}$$

$$a_d\left(L,\omega\right) \approx i\int_0^L \kappa(z,\omega)\exp\left[i\left(k_i(\omega)z + k_d(\omega)(L-z)\right)\right]dz\; a_i(0,\omega),$$

The spectrum of a diffracted signal at the output of the acousto-optic element can be approximated by a *product of the spectrum of an input signal with a transfer function*:

$$a_d(L,\omega) \approx H(\omega)\, a_i(0,\omega),$$

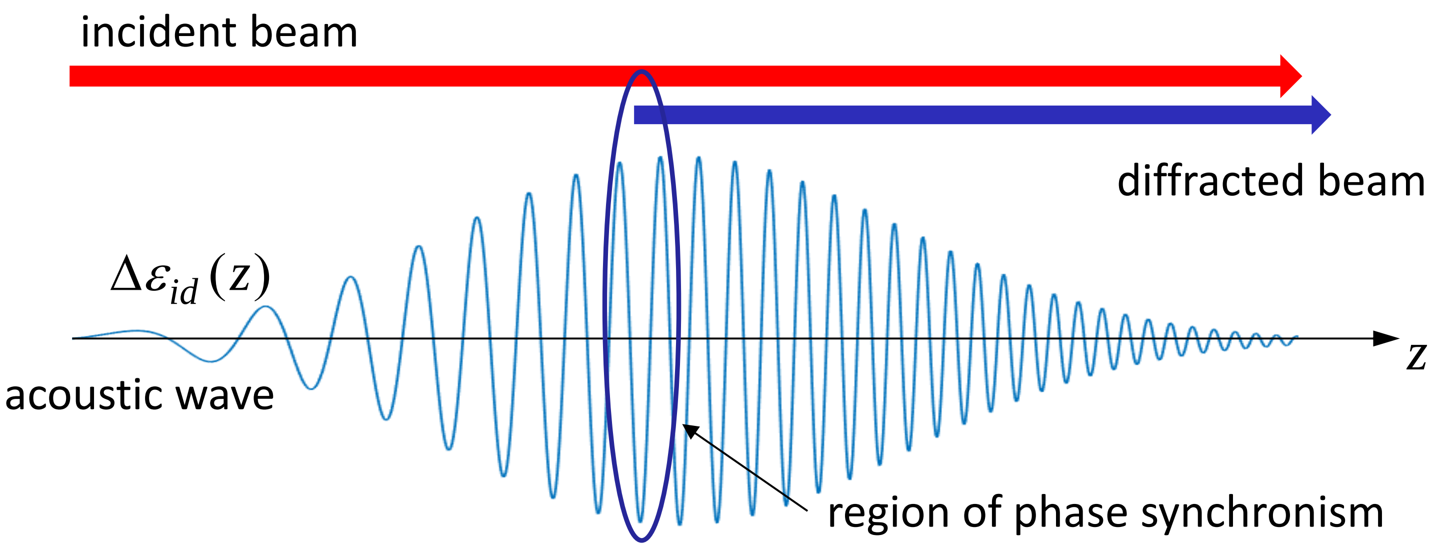
$$H(\omega) \approx i\int_0^L \kappa(z,\omega)\exp\Big[i\Big(k_i(\omega)z + k_d(\omega)(L-z)\Big)\Big]dz.$$

↑ incident wave dispersion ($k_i(\omega)$)

↑ diffracted wave dispersion ($k_d(\omega)$)

Inserting expression for $\Delta\varepsilon(z)$,

$$H(\omega) \approx i\frac{\omega}{4c}\exp\Big(ik_d(\omega)L\Big)\int_0^L \Delta\varepsilon_{id}(z)\,\underbrace{e^{i\left[k_i(\omega)-k_d(\omega)+K_a(z)\right]z}}_{\substack{\text{significat contribution only if}\\ k_i(\omega)-k_d(\omega)+K_a(z')\approx 0}}\,dz.$$



FASTLITE

Dazzler

Ultrafast pulse shaper

Dazzlers (or AOPDF) products are turn-key ultrafast pulse shaping systems, performing simultaneous and independent spectral phase and amplitude programming of ultrafast laser pulses.

With over 500 systems installed worldwide, the Dazzler is the reference tool for your pulse shaping applications.

Attention: there is Dazzler and Dazzler…!

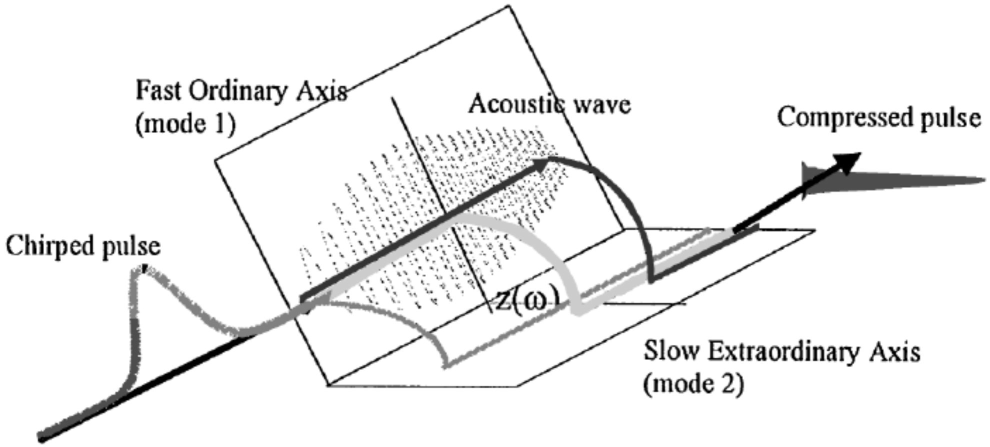
Dazzler

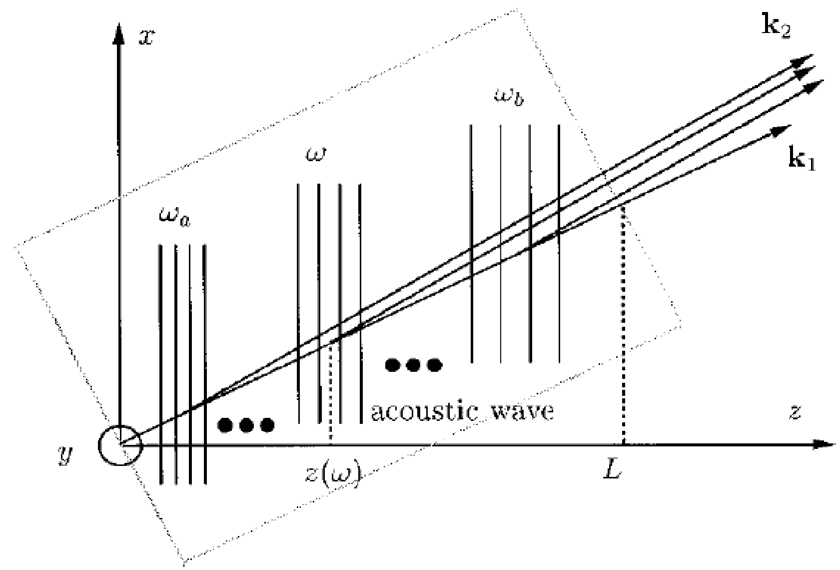
Weapon

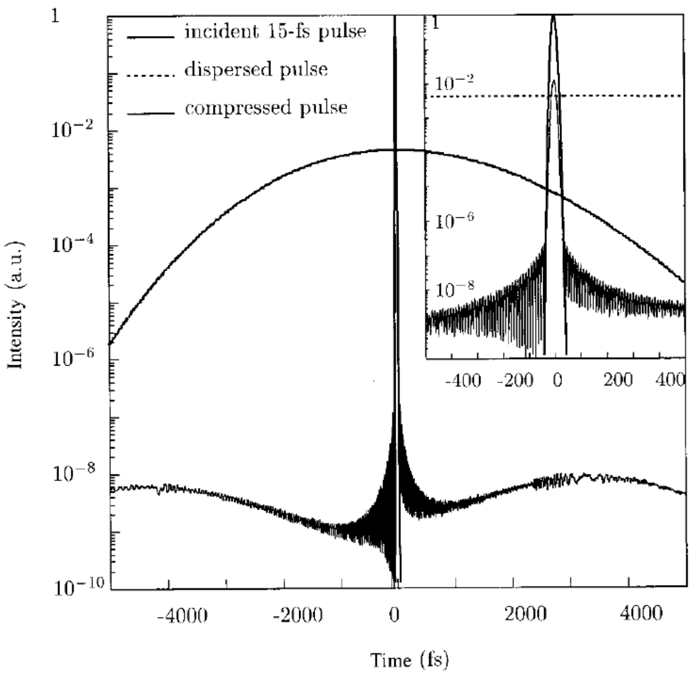
A dazzler is a non-lethal weapon which uses intense directed radiation to temporarily disable its target with flash blindness. Targets can include sensors or human vision. Initially developed for military use, non-military products are becoming available for use in law enforcement and security. Wikipedia

Acousto-optic dispersion control of ultrashort optical pulses (Dazzler)

Kvazikolineární interakce v TeO_2







$$E_{dif}(\omega) \sim E_{inc}(\omega)\cdot S(\omega \Delta n v_a / c)$$

$$H_d(t) \sim H_i(t) * V\left(ct/(\Delta n v_a)\right)$$

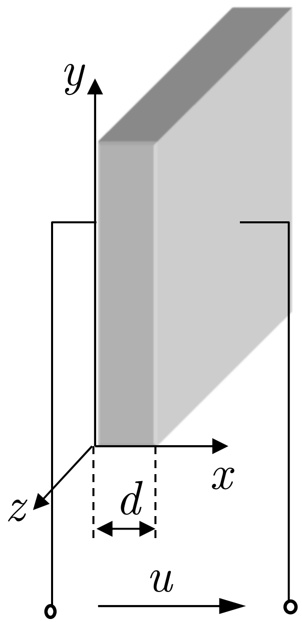
obecný lineární filtr;
vhodnou volbou časového průběhu ovládacího napětí piezoelektrického měniče je možné v širokých mezích ovládat časový průběh difraktovaného záření

F. Verluise, V. Laude, J.-P. Huignard, P. Tournois, A. Migus, J. Opt. Soc. Am. B, Vol. **17**, 138-145, 2000

F. Verluise, V. Laude, Z. Cheng, Ch. Spielmann, P. Tournois, Optics Letters Vol. **25**, 575-577, 2000

Excitation of acoustic waves by a piezoelectric transducer (1)

Transducer as an acoustic resonator



Relations in a piezoelectric medium:

$$\bar{\mathbf{T}} = \bar{\bar{\mathbf{c}}}^D : \bar{\mathbf{S}} - \tilde{\mathbf{h}} \cdot \mathbf{D},$$

$$\mathbf{E} = -\tilde{\mathbf{h}} : \bar{\mathbf{S}} + \bar{\boldsymbol{\eta}}^S \cdot \mathbf{D},$$

$$\bar{\boldsymbol{\eta}}^S = \left(\bar{\boldsymbol{\varepsilon}}^S\right)^{-1}, \quad \tilde{\boldsymbol{h}} = \bar{\boldsymbol{\eta}}^S \cdot \tilde{\boldsymbol{e}},$$

$$c^D_{jklm} = c^E_{jklm} + \eta^S_{pr} e_{pjk} e_{rlm},$$

$\bar{\mathbf{c}}^D = \bar{\mathbf{c}}^E + \tilde{\mathbf{e}} \cdot \bar{\boldsymbol{\eta}}^S \cdot \tilde{\mathbf{e}}$ is a „piezoelectrically stiffened“ tensor of elastic constants,

$\tilde{\mathbf{e}}$ is a piezoelectric tensor (of the 3rd rank).

If the tensor $\bar{\boldsymbol{\eta}}^S$ is diagonal, both $\mathbf{E}$ and $\mathbf{D}$ are oriented in the x-direction.

Let us introduce the "acoustic velocity" $\mathbf{u} = \dfrac{d\boldsymbol{\xi}}{dt}$. Then $\dfrac{dS_{jk}}{dt} = \dfrac{1}{2}\left(\dfrac{du_j}{dx_k} + \dfrac{du_k}{dx_j}\right)$.

The time derivative of the above equations can be supplemented with the Newton's force equation; we thus obtain the following set of equations:

Excitation of acoustic waves by a piezoelectric transducer (2)

$$\frac{\partial T_\alpha}{\partial t} = c^D_{\alpha\alpha}\frac{\partial u_l}{dx} - h_{1\alpha}\frac{\partial D_1}{\partial t},$$
$$\frac{\partial E_1}{\partial t} = -h_{1\alpha}\frac{\partial u_l}{\partial x} + \eta^S_{11}\frac{\partial D_1}{\partial t}, \quad \alpha = \begin{cases} 1 = (11) \text{ for } l = 1, \\ 6 = (12) \text{ for } l = 2, \\ 5 = (13) \text{ for } l = 3. \end{cases}$$
$$\frac{\partial T_a}{\partial x} = \rho\frac{\partial u_l}{\partial t},$$

l determines the direction of elastic deviation ("polarization" of the acoustic wave)

In case of harmonic time dependence $e^{j\Omega t}$ of all quantities, the equations can be integrated from $x = 0$ to $x = d$ with boundary conditions

$T_\alpha(0) = T_1,\ T_\alpha(d) = T_2,\ u_l(0) = u_1,\ u_l(d) = u_2.$

After some derivations we arrive to the set of equations

$$T_1 = \frac{Z_a}{j\tan\theta}u_1 + \frac{Z_a}{j\sin\theta}u_2 + \frac{h_{1\alpha}}{i\Omega}J,$$
$$T_2 = \frac{Z_a}{j\sin\theta}u_1 + \frac{Z_a}{j\tan\theta}u_2 + \frac{h_{1\alpha}}{j\Omega}J,$$
$$J = j\Omega\varepsilon^S_{11}E_1 - h_{1\alpha}\varepsilon^S_{11}u_1 - h_{1\alpha}\varepsilon^S_{11}u_2,$$

$Z_a = \sqrt{c^D_{\alpha\alpha}\rho} = \rho v_a$ … acoustic impedance,

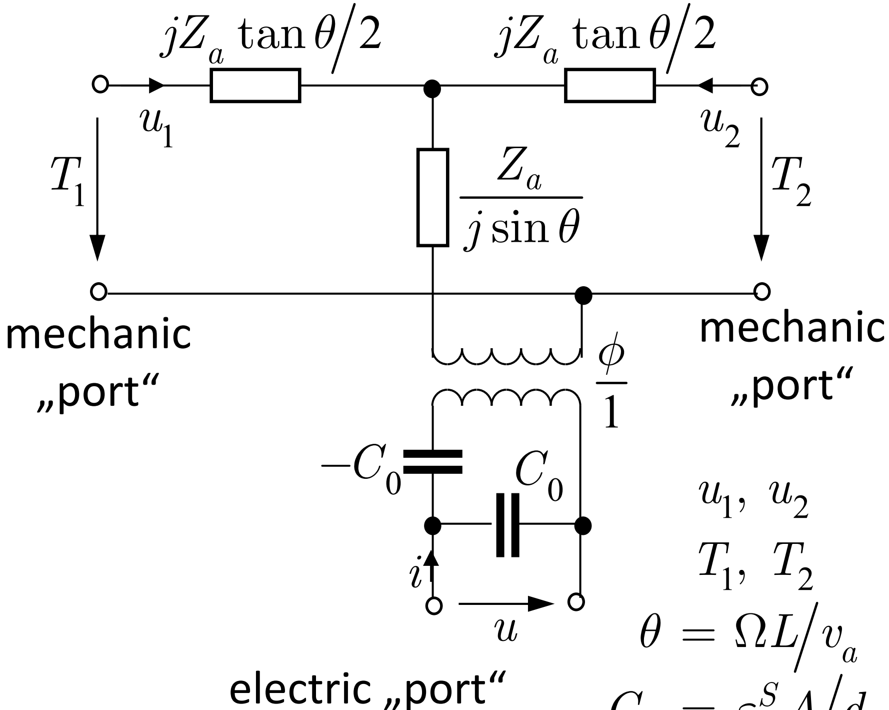
$J = j\Omega D_1$ … density of an induction electric current,

$\theta = \Omega d / v_a = K_a d = \pi\Omega / \Omega_0,$

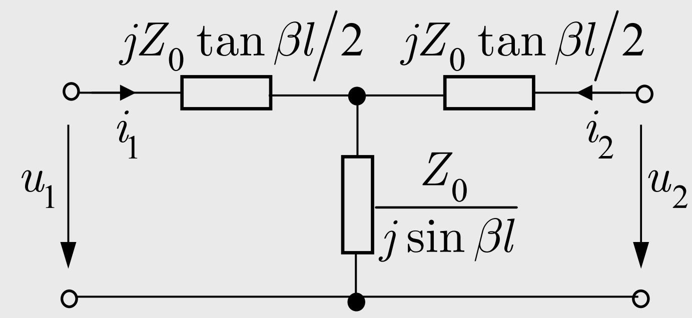
$f_0 = \Omega_0 / 2\pi = v_a / 2d$ …half-wave acoustic frequency

Excitation of acoustic waves by a piezoelectric transducer (3)

The last equations can be physically represented as an electric circuit –>
Mason's equivalent electric circuit of a piezoelectric transducer



Equivalent circuit of a section of a transmission line



$u_1,\ u_2$ … acoustic velocity at port 1, 2

$T_1,\ T_2$ … stress component at port 1, 2

$\theta = \Omega L / v_a$ …phase shift of acoustic wave

$C_0 = \varepsilon_{11}^S A / d,\ A\ldots$ transducer area,

$i = A.J$ … electric current / density,

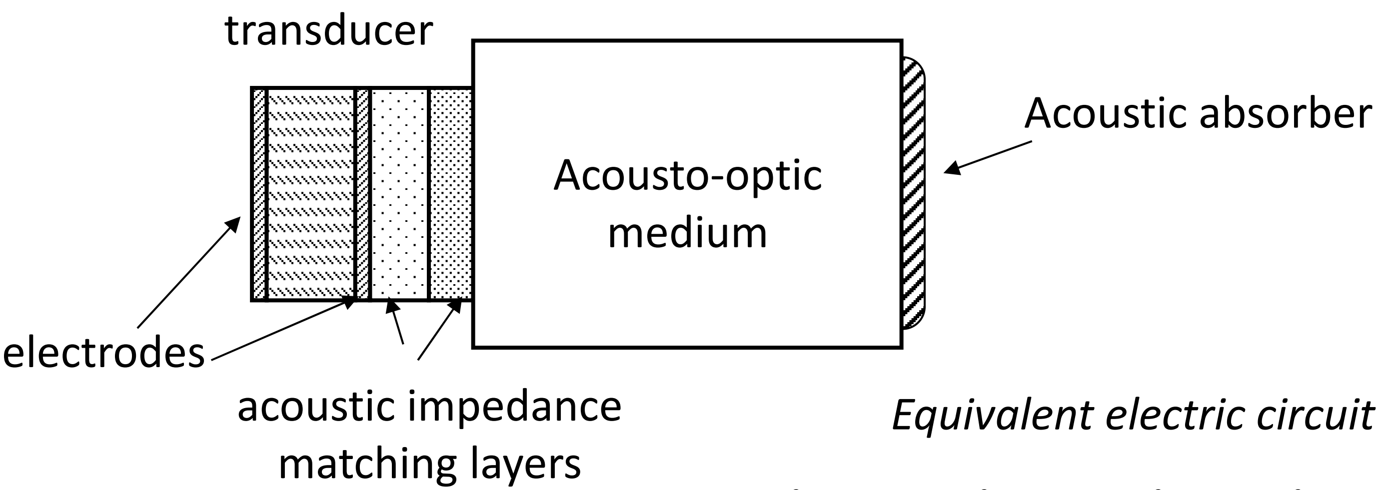
$u = E_1 d$ … electric voltage on the transducer,

$\phi = k\sqrt{Z_a \Omega_0 C_0 / \pi A}$ ...transformer conversion ratio

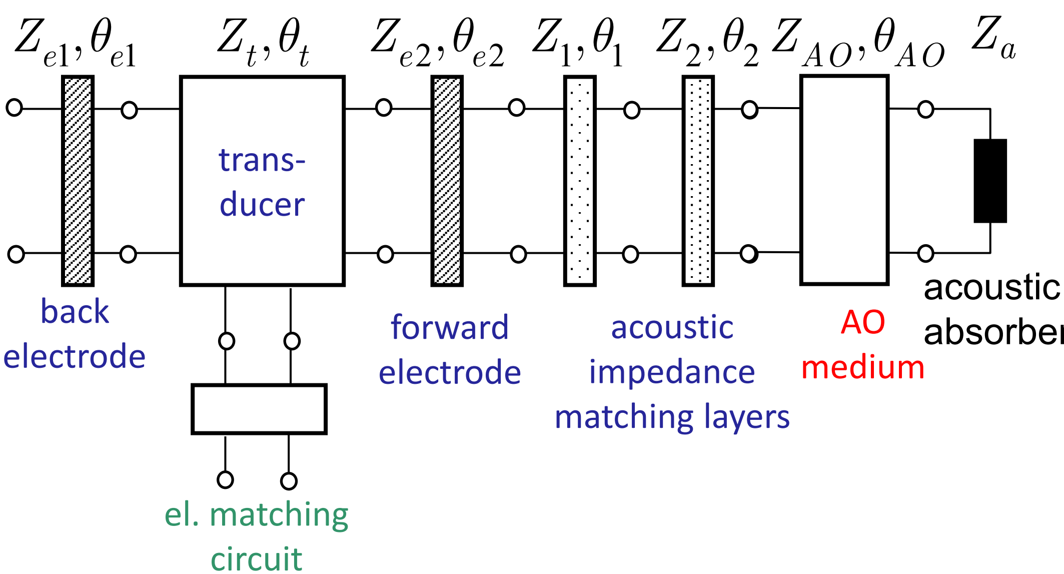
$k = h_{1\alpha} / \sqrt{\eta_{11}^S c_{\alpha\alpha}^D}$ …electro-mechanical coupling coefficient

Excitation of acoustic waves by a piezoelectric transducer (4)

(Frequency) transfer function of a piezoelectric transducer bonded to an acousto-optic medium can be analyzed and designed with the help of standard methods of the electric circuit theory



Equivalent electric circuit



Resulting frequency characteristics of and AO device

$$\left|F_{AO}(\Omega)\right|^2 = \left|F_{el}(\Omega)\right|^2 \cdot \left|F_{em}(\Omega)\right|^2 \cdot \eta_{AO}(\Omega)$$

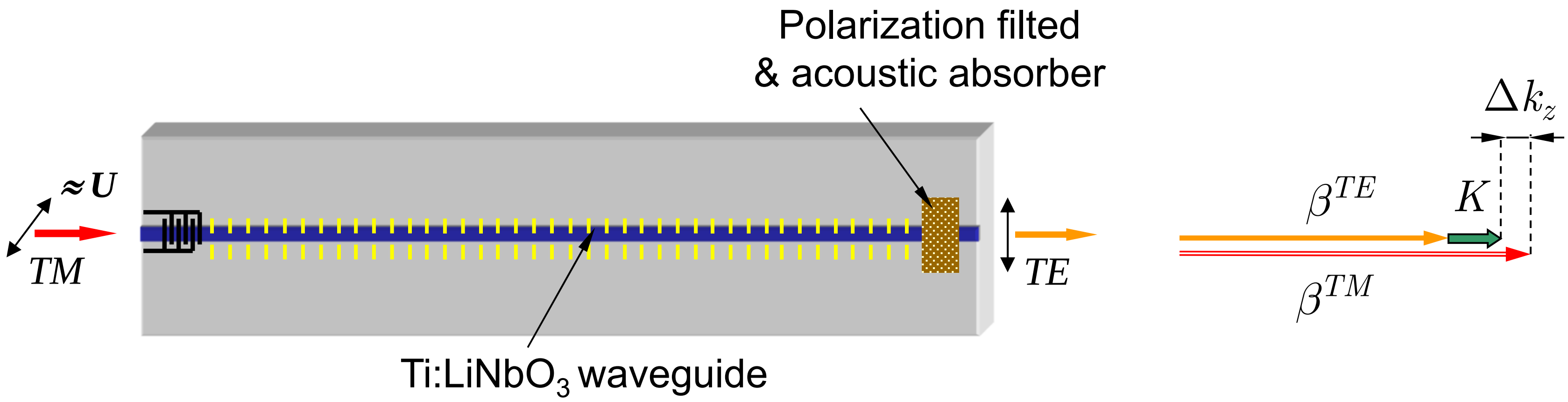
electric matching circuit | transducer +matching layers | acousto-optic diffraction efficiency

Selected piezoelectric and acoustic impedance matching materials

Material	ρ (g/cm^3)	ac. Mode	orientation	k	ε_r	v_a (km/s)	$Z_a = \rho \cdot v_a$
α-SiO_2 32	2.65	L S	X Y	0.098 0.137	4.58 4.58	5.75 3.85	15.2 10.2
$LiNbO_3$ 3m	4.64	L S	36°Y 163°Y	0.49 0.62	38.6 42.9	7.4 4.56	33.9 20.8
$LiTaO_3$ 3m	7.45	L S	47°Y X	0.29 0.44	42.7 42.6	7.2 4.22	55.2 31.4
ZnO 6mm	5.68	L S	Z 43°Y	0.282 0.322	8.84 8.63	6.40 3.21	36.4 18.4
In	7.3	L S				2.3 1.44	16.8 10.5
Au	19	L S				3.24 1.20	62.5 22.8
Ag	10.5	L S				3.65 1.61	38.0 16.7
Sn	7.2	L S				3.32 1.67	23.9 12.0

Integrated-optic acousto-optic devices

Collinear interaction



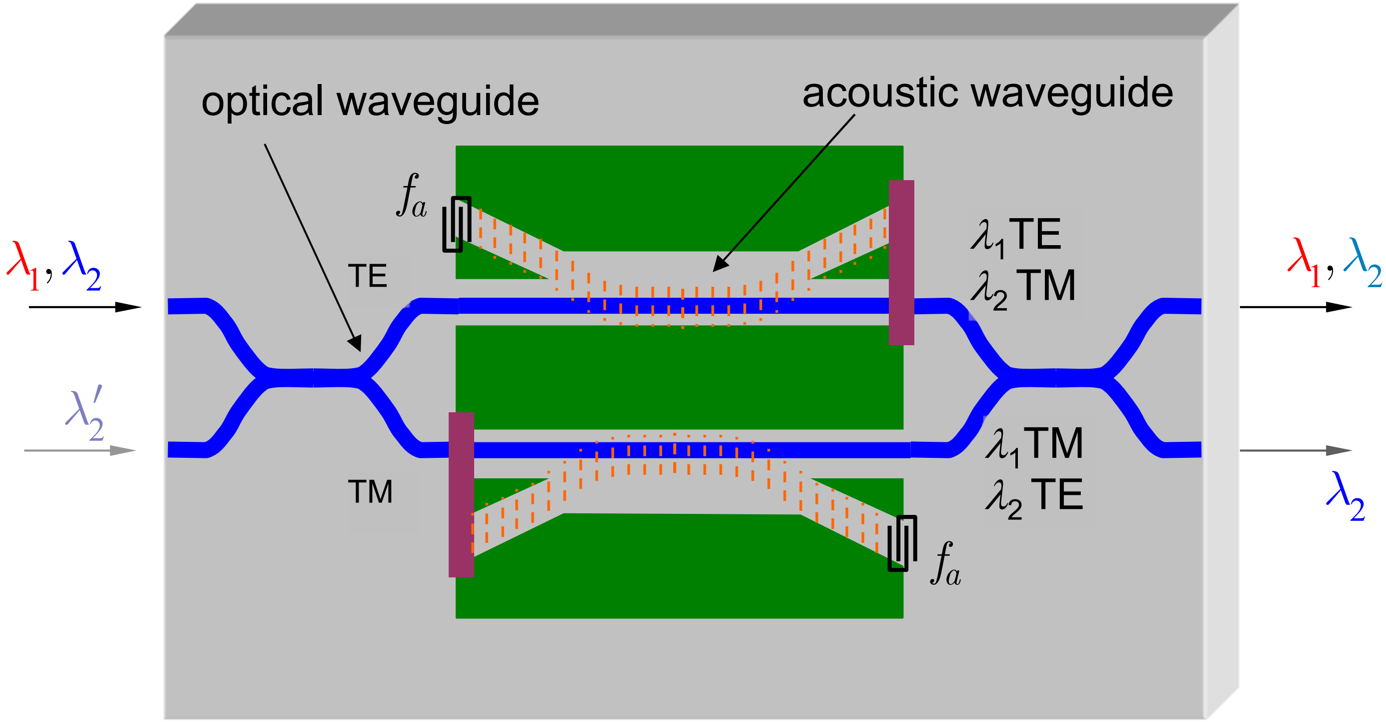
Efficiency of acousto-optic interaction

$$\eta = \frac{\kappa^2}{\kappa^2 + (\Delta k_z/2)^2} \sin^2\left(\sqrt{\kappa^2 + (\Delta k_z/2)^2} L\right),$$

$$\kappa \sim \frac{1}{2k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{e}^{TM}(x,y) \cdot \Delta\overline{\varepsilon}(x,y) \cdot \overline{e}^{TE}(x,y)\, dxdy = \frac{\pi}{2L_c}$$

Polarization-independent acousto-optic tuneable add-drop demultiplexor on $LiNbO_3$

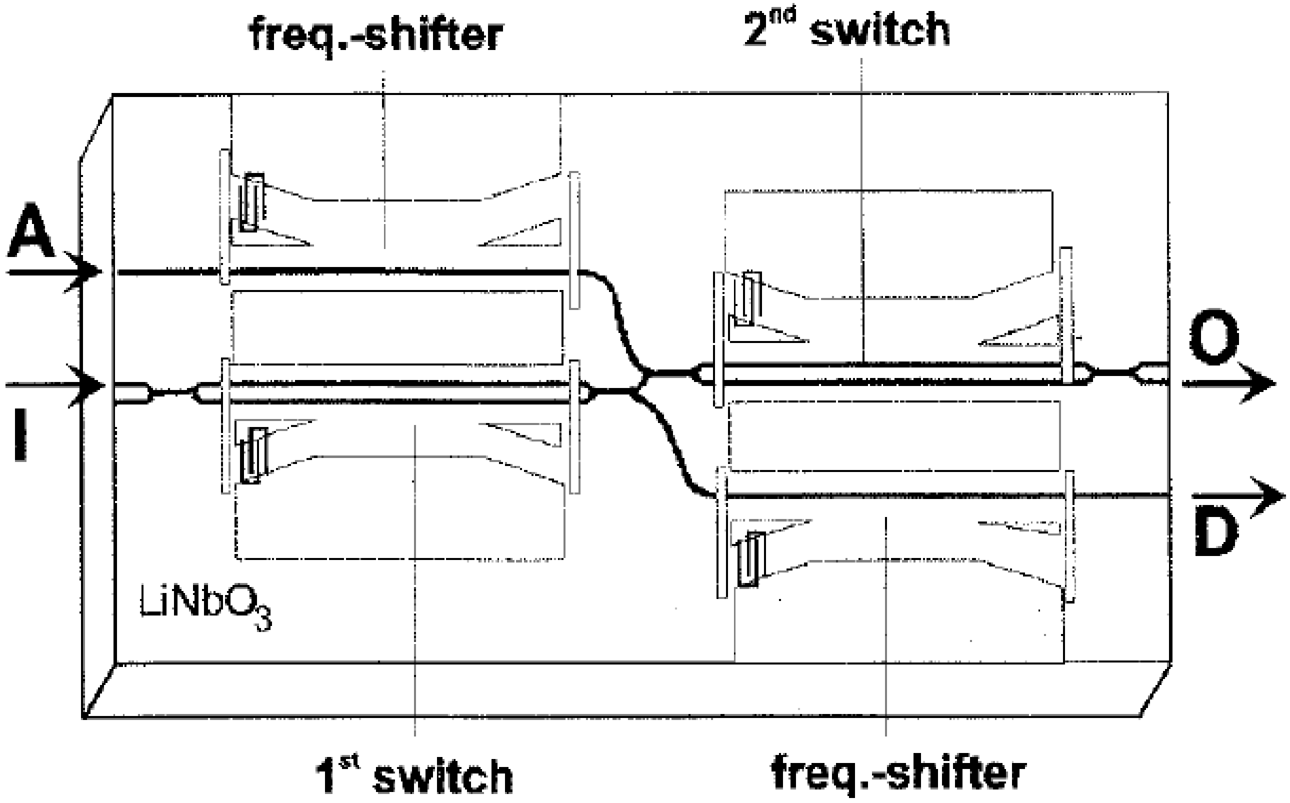
Principle: collinear AO TE-TM conversion



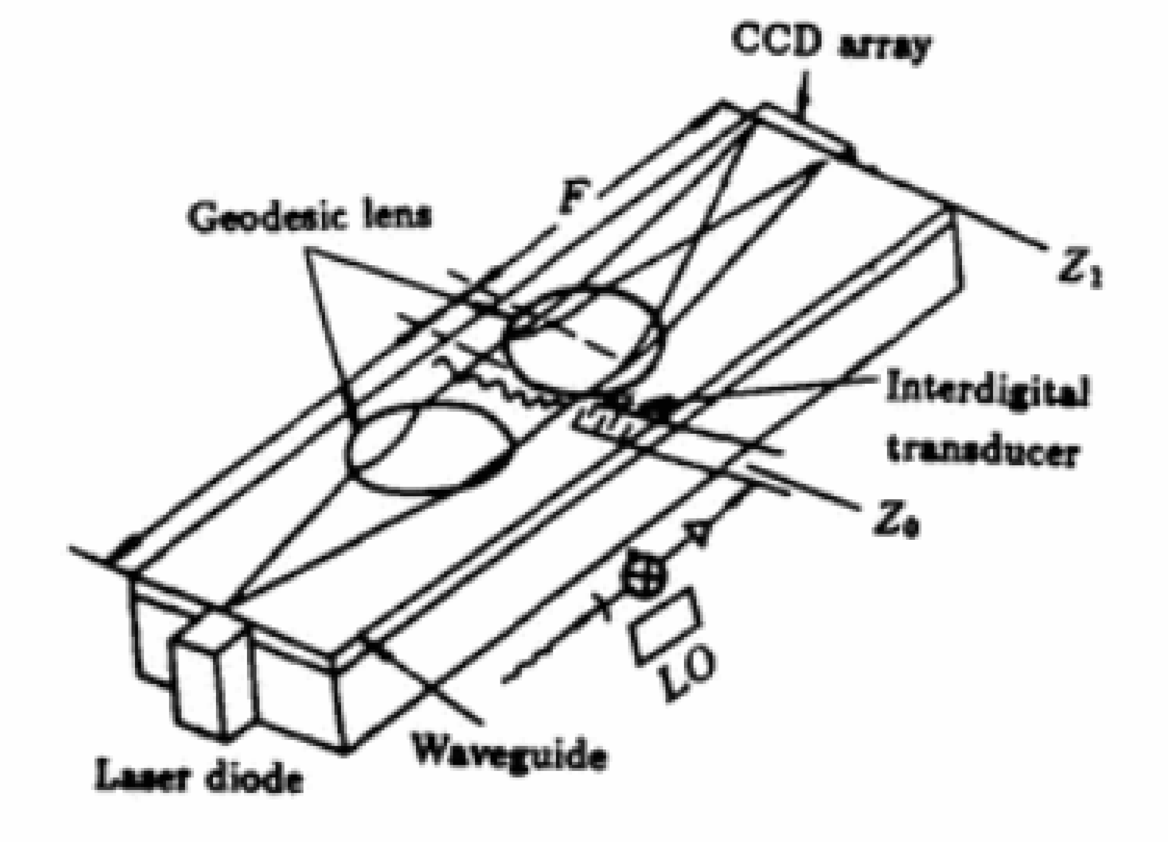
Average wavelength λ_c = 1,55 μm,
channel spacing < 1 nm, tuneability $\Delta\lambda \approx$ 70 nm

Frequency shift compensation

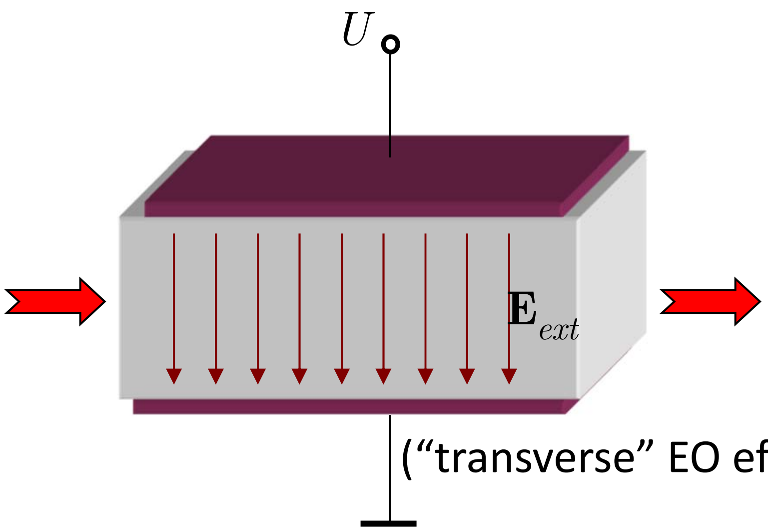
(Uni Paderborn, Germany, ECOC 1997)



Integrated-optical acousto-optic spectrum analyzer of RF signals



Theoretical fundamentals of electro-optic effect



Electric field applied to a material may change its optical properties (impermittivity).

If the magnitude of this change is linearly dependent on the applied field intensity, it is called *linear (Pockels) electro-optic effect*.

If the material response is quadratically dependent on the applied field intensity, it is called *quadratic (Kerr) electro-optic effect.*

Linear (Pockels) effect can take place only in materials, the physical properties of which are sensitive to the inversion of coordinates (non-centrosymmetric materials).

$$\mathbf{D} = \varepsilon_0 \overline{\varepsilon} \cdot \mathbf{E}, \qquad \mathbf{E} = \frac{1}{\varepsilon_0} \overline{\eta} \cdot \mathbf{D}, \qquad \overline{\eta} = \overline{\varepsilon}^{-1}.$$

$$\overline{\eta}\left(\mathbf{E}_{ext}\right) = \overline{\eta}\left(\mathbf{0}\right) + \underbrace{\tilde{\boldsymbol{r}} \cdot \mathbf{E}_{ext}}_{\text{linear effect (Pockels)}} + \underbrace{\overline{\overline{\boldsymbol{s}}} : \mathbf{E}_{ext}\mathbf{E}_{ext}}_{\text{quadratic effect (Kerr)}}$$

$$\overline{\eta} + \Delta\overline{\eta} = \left(\overline{\varepsilon} + \Delta\overline{\varepsilon}\right)^{-1};$$

$$\left(\overline{\eta} + \Delta\overline{\eta}\right) \cdot \left(\overline{\varepsilon} + \Delta\overline{\varepsilon}\right) = \overline{\mathbf{I}};$$

$$\underbrace{\overline{\eta} \cdot \overline{\varepsilon}}_{\overline{\mathbf{I}}} + \overline{\eta} \cdot \Delta\overline{\varepsilon} + \Delta\overline{\eta} \cdot \overline{\varepsilon} + \underbrace{\Delta\overline{\eta} \cdot \Delta\overline{\varepsilon}}_{\text{small term of } 2^{nd} \text{ order}} = \overline{\mathbf{I}};$$

$$\overline{\eta} \cdot \Delta\overline{\varepsilon} = -\Delta\overline{\eta} \cdot \overline{\varepsilon};$$

$$\Delta\overline{\varepsilon} = -\overline{\eta}^{-1} \cdot \Delta\overline{\eta} \cdot \overline{\varepsilon};$$

$$\boxed{\Delta\overline{\varepsilon} = -\overline{\varepsilon} \cdot \Delta\overline{\eta} \cdot \overline{\varepsilon}}$$

Theoretical fundamentals of electro-optic effect

Since the tensor $\overline{\boldsymbol{\varepsilon}}$ is symmetric, $\overline{\boldsymbol{\eta}} = \overline{\boldsymbol{\varepsilon}}^{-1}$ is also symmetric. The tensor $\overline{\boldsymbol{r}}$ is thus invariant with respect to the interchange of the *first two* subscripts ("belonging to" $\overline{\boldsymbol{\eta}}$):

$$r_{jkl} = r_{kjl}.$$

Similarly, the tensor $\overline{\overline{\boldsymbol{s}}}$ is invariant with respect to the interchange of the first two and the second two subscripts:

$$s_{jklm} = s_{kjlm} = s_{jkml} = s_{kjml}.$$

We can thus apply the shortened Voight notation

$r_{jkl} \Rightarrow r_{\alpha l}, \qquad s_{jklm} = s_{\alpha\beta},$
$\alpha, \beta = 1,\ 2,\ 3$ for $(11),(22),(33)$, and
$\alpha, \beta = 4,\ 5,\ 6$ for $(23) \equiv (32),(13) \equiv (31),(12) \equiv (21)$.

Since $\Delta\overline{\eta} = \tilde{\boldsymbol{r}} \cdot \mathbf{E}_{ext} + \overline{\overline{\boldsymbol{s}}} : \mathbf{E}_{ext}\mathbf{E}_{ext}$, the physical units of these tensors are

$\left[\tilde{r}\right] =$ m/V (in reality often pm/V) , $\qquad \left[\overline{\overline{s}}\right] = \mathrm{m}^2/\mathrm{V}^2$.

Properties of some important electro-optic materials (1)

Dielectric crystals of the group ADP
Grown from water solution; large but hygroscopic
Point symmetry group $\bar{4}2m$, uniaxial anisotropic crystals

ADP: $r_{41} = 23.11$ pm/V, $n_o = 1.522$
$r_{63} = 8.5$ pm/V, $n_e = 1.4773$

KDP: $r_{41} = 8$ pm/V, $n_o = 1.5074$
$r_{63} = 11$ pm/V, $n_e = 1.4661$

DKDP: $r_{41} = 26$ pm/V, $n_o = 1.502$
$r_{63} = 24.1$ pm/V, $n_e = 1.462$

$$\left(r_{\alpha j}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{pmatrix}$$

Properties of some important electro-optic materials (2)

Semiconductor crystals of the group $A^{III}B^{V}$ (GaAs, InP)
Point symmetry group $\bar{4}3m$, *isotropic materials!*

GaAs: $\lambda = 1.15\ \mu\text{m}$

$r_{41} = 1.43\ \text{pm/V}, \qquad n = 3.43$

$$\left(r_{\alpha j}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{41} \end{pmatrix}$$

Ferroelectric crystals $LiNbO_3$ and $LiTaO_3$
Point symmetry group $3m$, uniaxial crystals; high LF permittivity

$LiNbO_3$: $r_{22} = 6.8\ \text{pm/V}, \qquad n_o = 2.286$

$r_{13} = 10\ \text{pm/V}, \qquad n_e = 2.202$

$r_{33} \doteq 30\ \text{pm/V},$

$r_{51} \doteq 32\ \text{pm/V}$

$LiTaO_3$: $\sim$ " $\qquad n_o = 2.176$

$n_e = 2.180$

$$\left(r_{\alpha j}\right) = \begin{pmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{pmatrix}$$

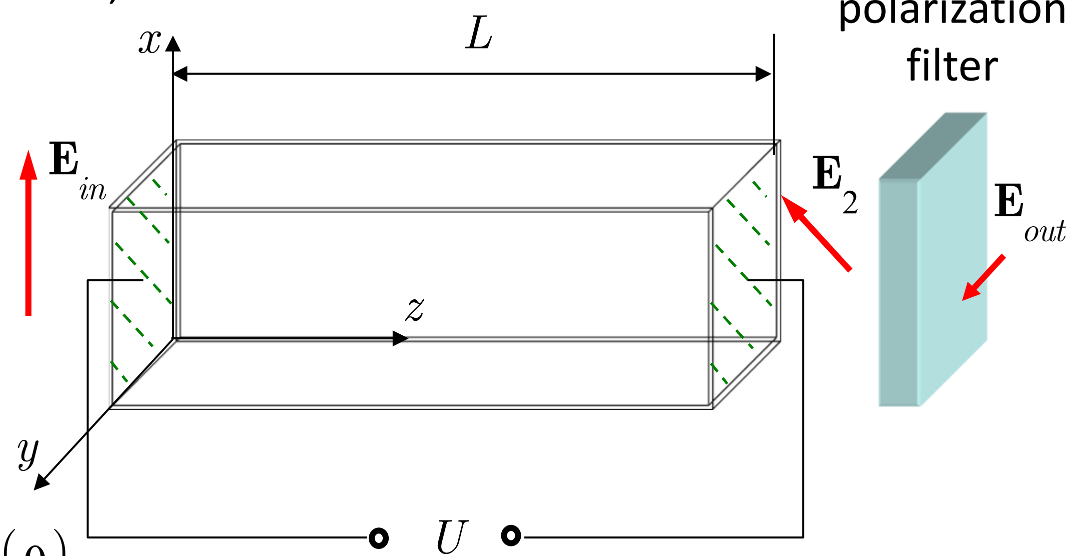
Optical wave propagation in a electrooptic material under external field (1)

Change of optical permittivity due to an applied field $\mathbf{E}_{ext}$: $\Delta\bar{\varepsilon} = -\bar{\varepsilon}\cdot\left(\tilde{r}\cdot\mathbf{E}_{ext}\right)\cdot\bar{\varepsilon};$

Fresnel dispersion equation is then $\left[\mathbf{l}^2\bar{\mathbf{I}} - \mathbf{ll} - \bar{\varepsilon} + \bar{\varepsilon}\cdot\left(\tilde{r}\cdot\mathbf{E}_{ext}\right)\cdot\bar{\varepsilon}\right]\cdot\mathbf{E}_0 = \mathbf{0}.$

Example 1: Amplitude modulator in KDP, Z-cut

$$\mathbf{l} = l_z\mathbf{z}^0;\ \mathbf{E}_{ext} = E_{ext}\mathbf{z}^0;$$
$$\tilde{r}\cdot\mathbf{E}_{ext} = r_{63}\left(\mathbf{x}^0\mathbf{y}^0 + \mathbf{y}^0\mathbf{x}^0\right)E_{ext};$$
$$\Delta\varepsilon_{12} = \Delta\varepsilon_{21} = -n_o^4 r_{63}E_{ext}.$$



("longitudinal" electro-optic effect)

Eigenwave equation is

$$\begin{pmatrix} l_z^2 - n_o^2 & n_o^4 r_{63}E_{v.} & 0 \\ n_o^4 r_{63}E_{v.} & l_z^2 - n_o^2 & 0 \\ 0 & 0 & -n_e^2 \end{pmatrix}\cdot\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Amplitude modulator on the Z-cut KDP

Dispersion equation: $\left(l_z^2 - n_o^2\right)^2 - \left(n_o^4 r_{63} E_{ext}\right)^2 = 0.$ The solution is

$$l_{z1,2} = \sqrt{n_o^2 \pm n_o^4 r_{63} E_{ext}} \cong n_o \pm \frac{1}{2} n_o^3 r_{63} E_{ext} = n_o \pm \Delta n,\ \Delta n = \frac{1}{2} n_o^3 r_{63} E_{ext}.$$

"Eigenwave" equation

$$\begin{pmatrix} l_z^2 - n_o^2 & n_o^4 r_{63} E_{ext} \\ n_o^4 r_{63} E_{ext} & l_z^2 - n_o^2 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$ or

$$\frac{E_y}{E_x} = -\frac{n_o^4 r_{63} E_{ext}}{l_z^2 - n_o^2} = \mp 1;$$

Eigenwaves are thus *linearly polarized waves, polarized* under angle of 45° with respect to the coordinate axes.

The input wave $\mathbf{E}_{in} = E_0 \mathbf{x}^0$ can be decomposed into two waves with polarizations $\mathbf{e}_{1,2}^0 = \frac{1}{\sqrt{2}}\left(\mathbf{x}^0 \pm \mathbf{y}^0\right)$ as follows: $\mathbf{E}_{in} = E_0 \mathbf{x}^0 = \frac{1}{\sqrt{2}} E_0 \left(\mathbf{e}_1^0 + \mathbf{e}_2^0\right).$

Its propagation by the distance *z* can then be described as

$$\mathbf{E}(z) = \frac{e^{ik_0 n_0 z}}{\sqrt{2}} E_0 \left(\mathbf{e}_1^0 e^{ik_0 \Delta n z} + \mathbf{e}_2^0 e^{-ik_0 \Delta n z}\right) = e^{ik_0 n_0 z} E_0 \left(\mathbf{x}^0 \cos k_0 \Delta n z + i \mathbf{y}^0 \sin k_0 \Delta n z\right).$$

At the output $z = L$ after the polarizer blocking the *x*-component we detect the intensity

$$I(L) = \left|E_y(L)\right|^2 = E_0^2 \sin^2(k_0 \Delta n L) = I(0) \sin^2(k_0 \Delta n L).$$

Amplitude modulator on the Z-cut KDP - continuation

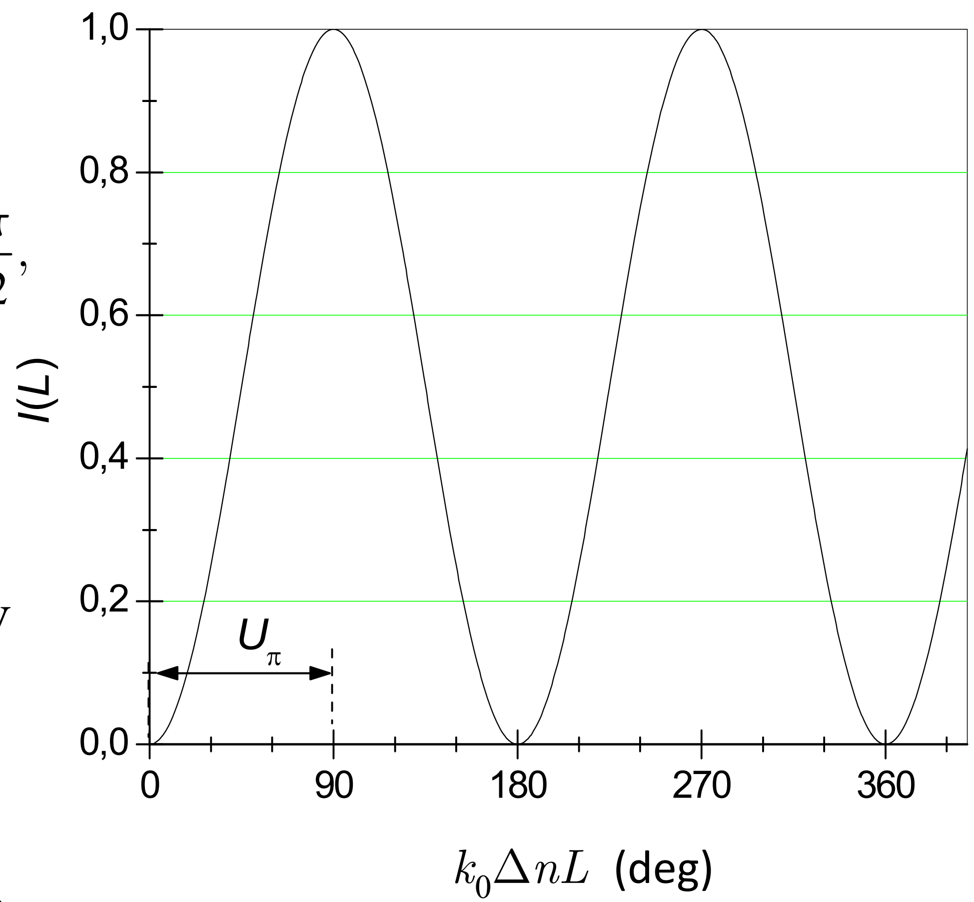
"Half-wave" voltage:

$$k_0 \Delta n L = \frac{\pi}{2}, \quad \text{i.e.} \quad \frac{1}{2} k_0 n_o^3 r_{63} E_{ext} L \overset{!}{=} \frac{\pi}{2},$$

$$U_\pi = E_{ex} L = \frac{\pi}{k_0 n_o^3 r_{63}} = \frac{\lambda}{2 n_o^3 r_{63}}$$

For KDP at 633 nm, $U_\pi \doteq 9.3\ \text{kV}$

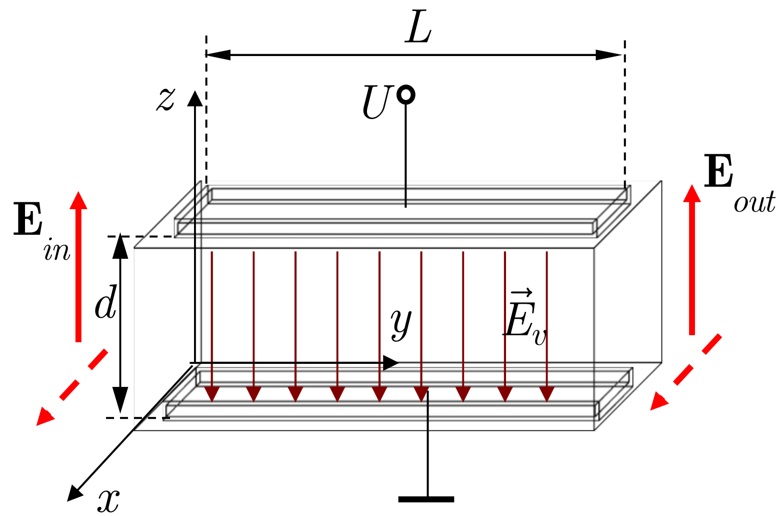
In case of the "longitudinal"
EO effect, the half-wave voltage
is fully determined by the parameters
of the EO material and on the wavelength.



Example 2: Phase modulator in $LiNbO_3$

("transverse" EO effect)

$$\mathbf{l} = l_y \mathbf{y}^0, \qquad \mathbf{E}_{ext} = E_v \mathbf{z}^0,$$



$$\Delta\overline{\varepsilon} = -\overline{\varepsilon} \cdot \left(\tilde{r} \cdot \vec{z}^0 E_{ext}\right) \cdot \overline{\varepsilon} =$$

$$= \begin{pmatrix} -n_o^4 r_{13} E_{ext} & 0 & 0 \\ 0 & -n_o^4 r_{13} E_{ext} & 0 \\ 0 & 0 & -n_e^4 r_{33} E_{ext} \end{pmatrix}.$$

$$\begin{pmatrix} l_y^2 - n_o^2 + n_o^4 r_{13} E_{ext} & 0 & 0 \\ 0 & -n_o^2 + n_o^4 r_{13} E_{ext} & 0 \\ 0 & 0 & l_y^2 - n_e^2 + n_e^4 r_{33} E_{ext} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . E_y \equiv 0$$

In dependence on the input polarization $\left(E_x \text{ or } E_z\right)$

$$\Delta n_o \doteq -\frac{1}{2} n_o^3 r_{13} E_{ext}, \quad \text{or} \quad \Delta n_e \doteq -\frac{1}{2} n_e^3 r_{33} E_{ext}.$$

Example 2: Phase modulator in $LiNbO_3$ - continuation

Phase change after passing the length *L* due to the applied voltage is

$$E_x(L) = E_0 e^{ik_0 n_o L} e^{-i\frac{1}{2}k_0 n_o^3 r_{13} E_{ext} L}, \quad \Delta\varphi_o = -\frac{1}{2}k_0 n_o^3 r_{13} L E_{ext},$$

$$E_z(L) = E_0 e^{ik_0 n_e L} e^{-i\frac{1}{2}k_0 n_e^3 r_{33} E_{ext} L}, \quad \Delta\varphi_e = -\frac{1}{2}k_0 n_e^3 r_{33} L E_{ext}.$$

Contrary to the case of a "longitudinal" effect, $E_v = U/d$, where d is the electrode separation.

The "half-wave voltage" is now defined as a voltage needed for the phase change by π:

$$U_\pi = E_{ext} d = \frac{\lambda}{n_o^3 r_{13}} \frac{d}{L} \quad \text{for } E_{in} = E_x,$$

$$U_\pi = E_{ext} d = \frac{\lambda}{n_e^3 r_{33}} \frac{d}{L} \quad \text{for } E_{in} = E_z.$$

For $LiNbO_3$

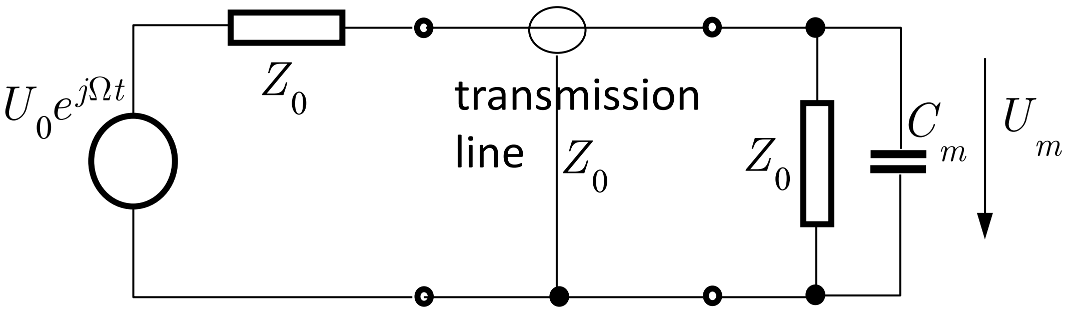
$$U_\pi \cong 5.3\frac{d}{L} \ \text{[kV] pro } E_x,$$

$$U_\pi \cong 2.2\frac{d}{L} \ \text{[kV] pro } E_z.$$

EO modulation speed (modulation bandwidth)

Intrinsic reaction time of an EO effect is very short, of the order of $10^{-14} \div 10^{-15}$ s. Limitation is due to the charging/discharging time of the electrode capacitance:

Exception: complex representation $e^{j\Omega t}$ (electronic signals)



Capacitance of electrodes

$$C_m = \frac{\varepsilon_0 \varepsilon_{el} A}{d} \approx 2 \div 5 \text{ pF}$$

Typical characteristic impedance of the feeding line is $Z_0 = 50\ \Omega$

Circuit transfer function:

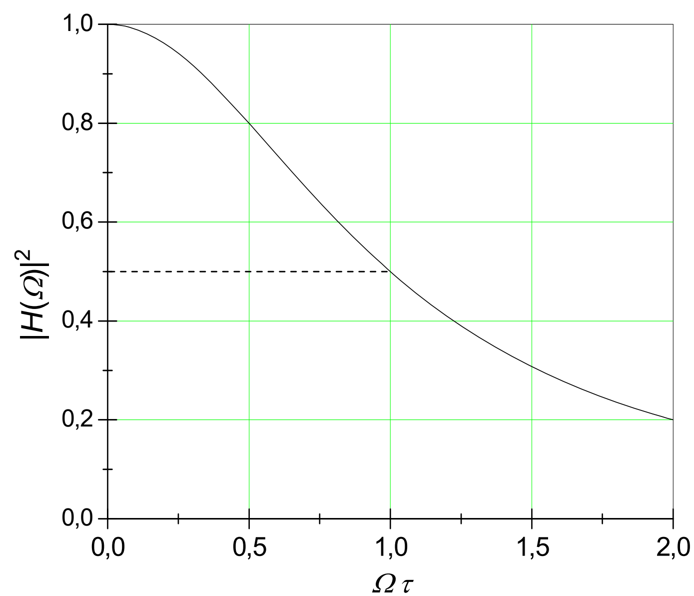
$$H(\Omega) = \frac{U_m}{U_0} = \frac{1}{2} \cdot \frac{1}{1 + j\Omega C_m Z_0 / 2} = \frac{1}{2} \cdot \frac{1}{1 + j\Omega\tau},$$

$$\tau = C_m Z_0 / 2.$$

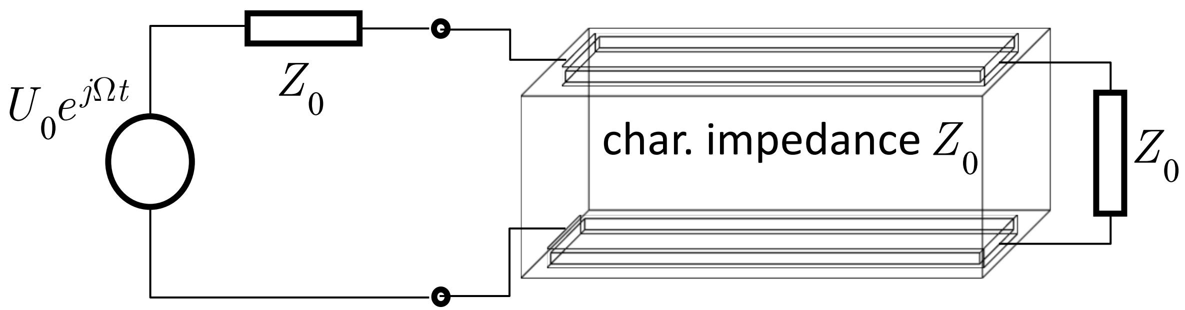
$$|H(\Omega)|^2 = \frac{1}{4} \cdot \frac{1}{1 + (\Omega\tau)^2}.$$

3 dB decrease takes place for $\Omega\tau \approx 1$, i.e.,

$$\Omega_{\max} \approx \frac{1}{\tau}, \qquad f_{\max} = B = \frac{1}{\pi C_m Z_0} \approx 3 \text{ GHz}.$$



Extending modulation bandwidth by using travelling-wave electrodes



Optical wave: $E_{opt} = E_0 \exp\left[j\left(\omega t - k_0 n_{eff} y\right)\right],$

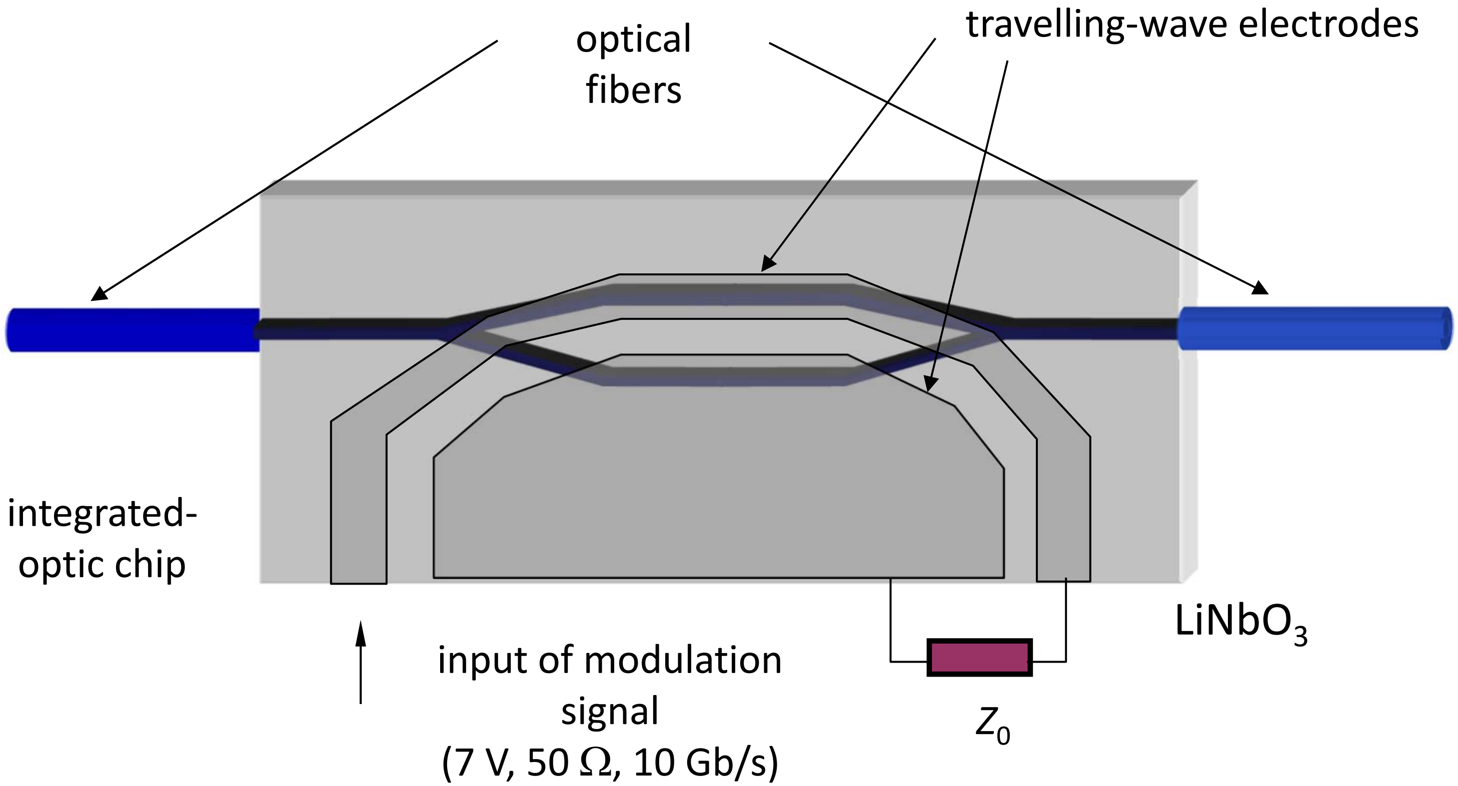
Modulation wave: $E_{\mathrm{mod}} = E_m \exp\left[j\left(\Omega t - k_0 n_m y\right)\right].$

It can be shown that the modulation efficiency is $\eta_{\mathrm{mod}} \sim \left[\dfrac{\sin \frac{\Omega}{2c}\left(n_m - n_{eff}\right)L}{\frac{\Omega}{2c}\left(n_m - n_{eff}\right)L} \right]^2;$

4-dB bandwidth-length product is now $B \cdot L \approx \dfrac{\Omega_{\max}}{2\pi} L = \dfrac{c}{2(n_m - n_{eff})} \approx 10\ \mathrm{GHz.cm}$

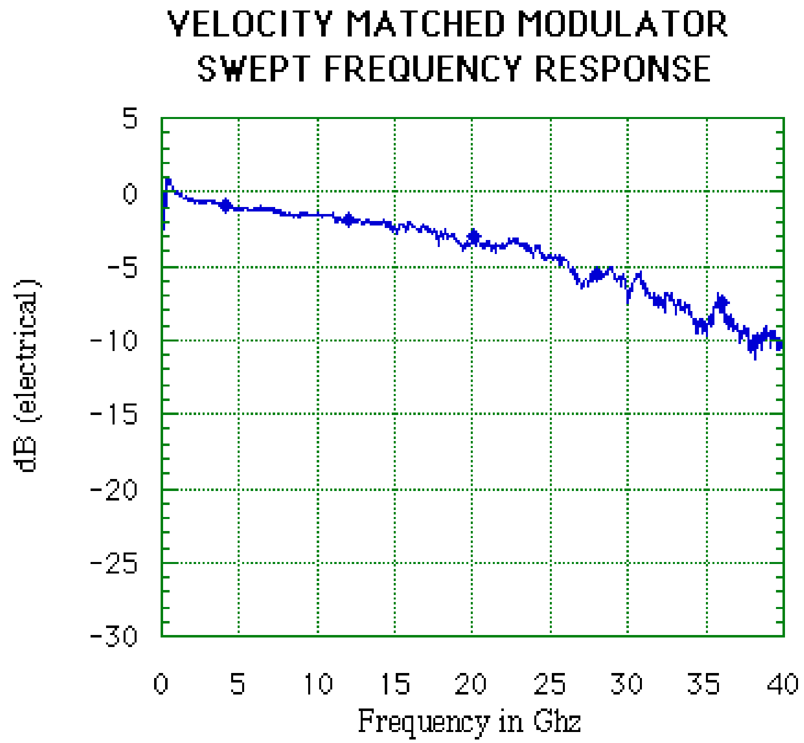
Elektro-optically controlled Mach-Zehnder interferometric modulator

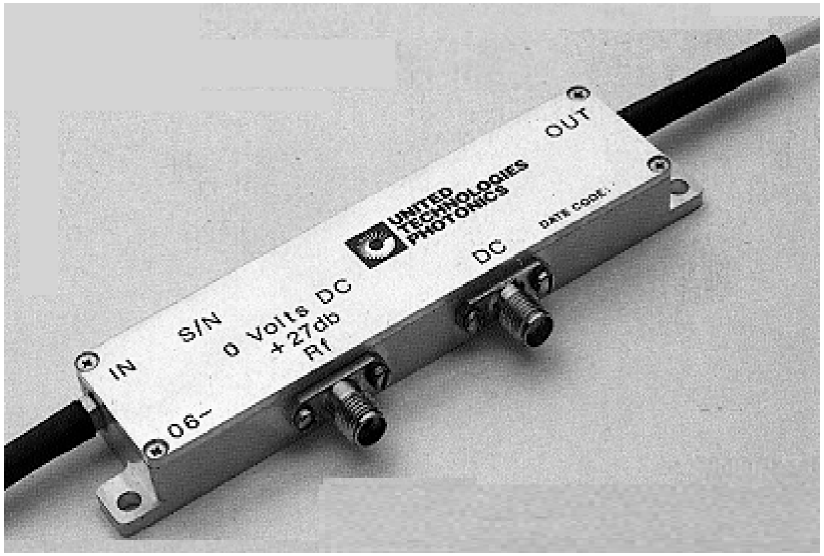
External modulator for optical communication systems with modulation bandwidth ≥ 10 Gb/s



Probably the most used electro-optic components

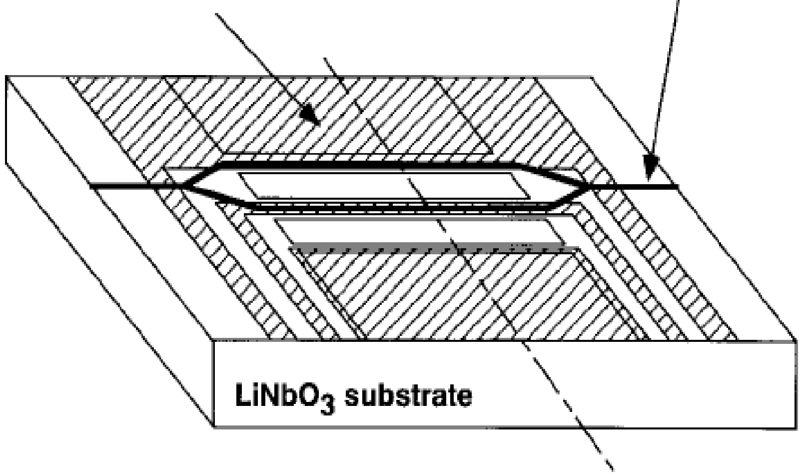
Electro-optic MZ modulator

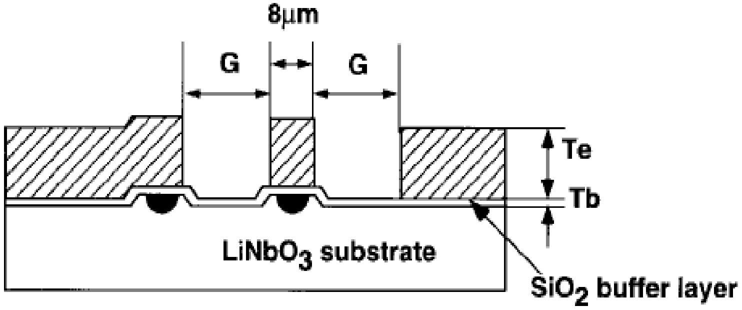


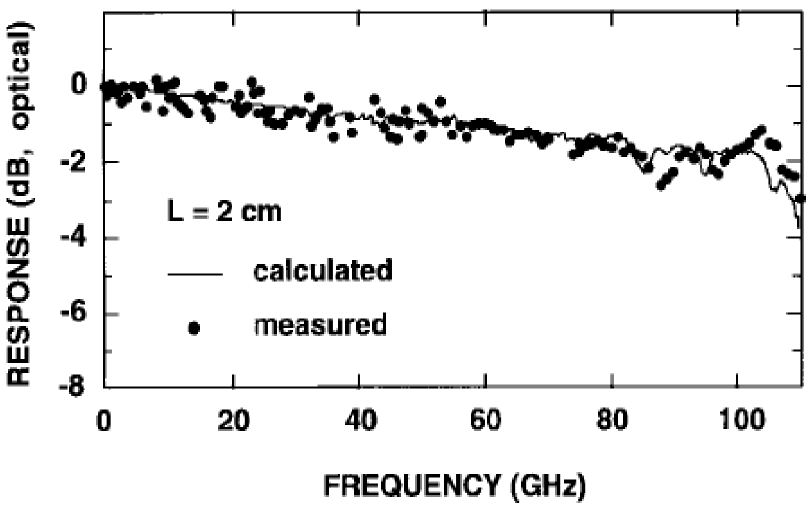


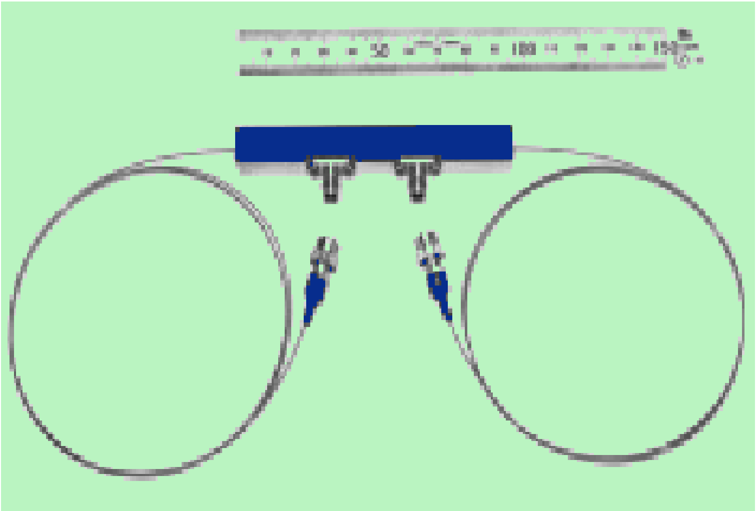
100 GHz $LiNbO_3$ modulator with half-wave voltage of 5.1 V

(NTT, 1998)

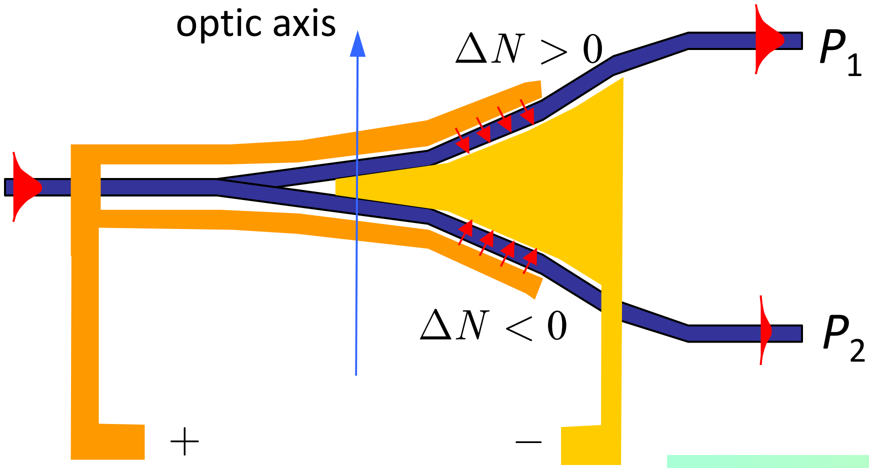






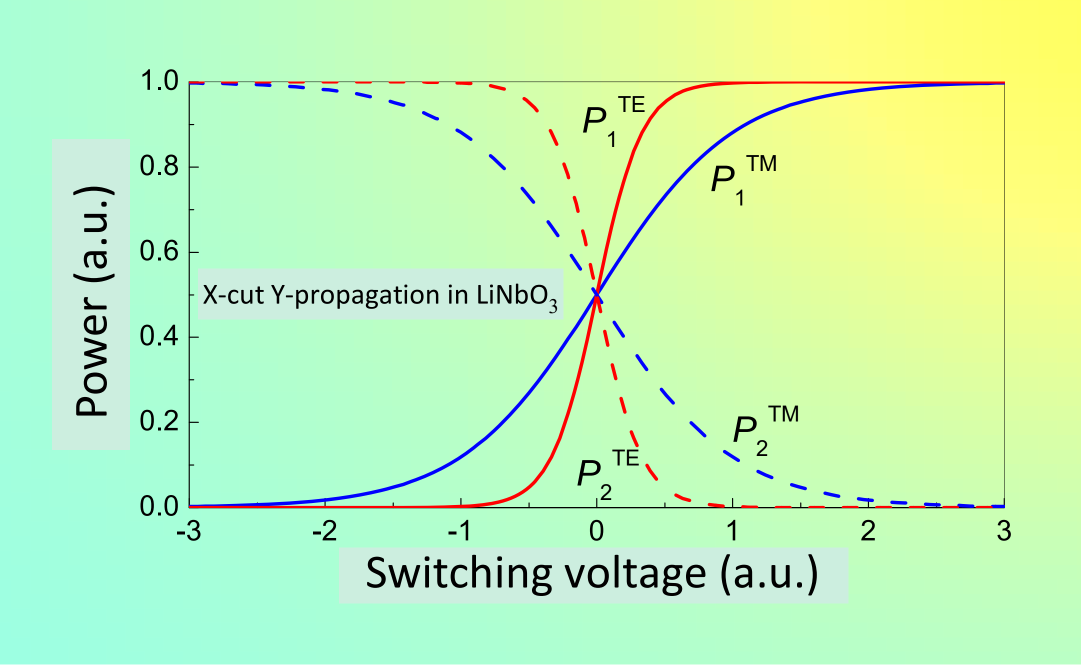


Polarization-independent „digital“ optic switch (DOS) in $LiNbO_3$



Symmetric Y-junction
with electrooptically induced
asymmetry

Switching characteristics



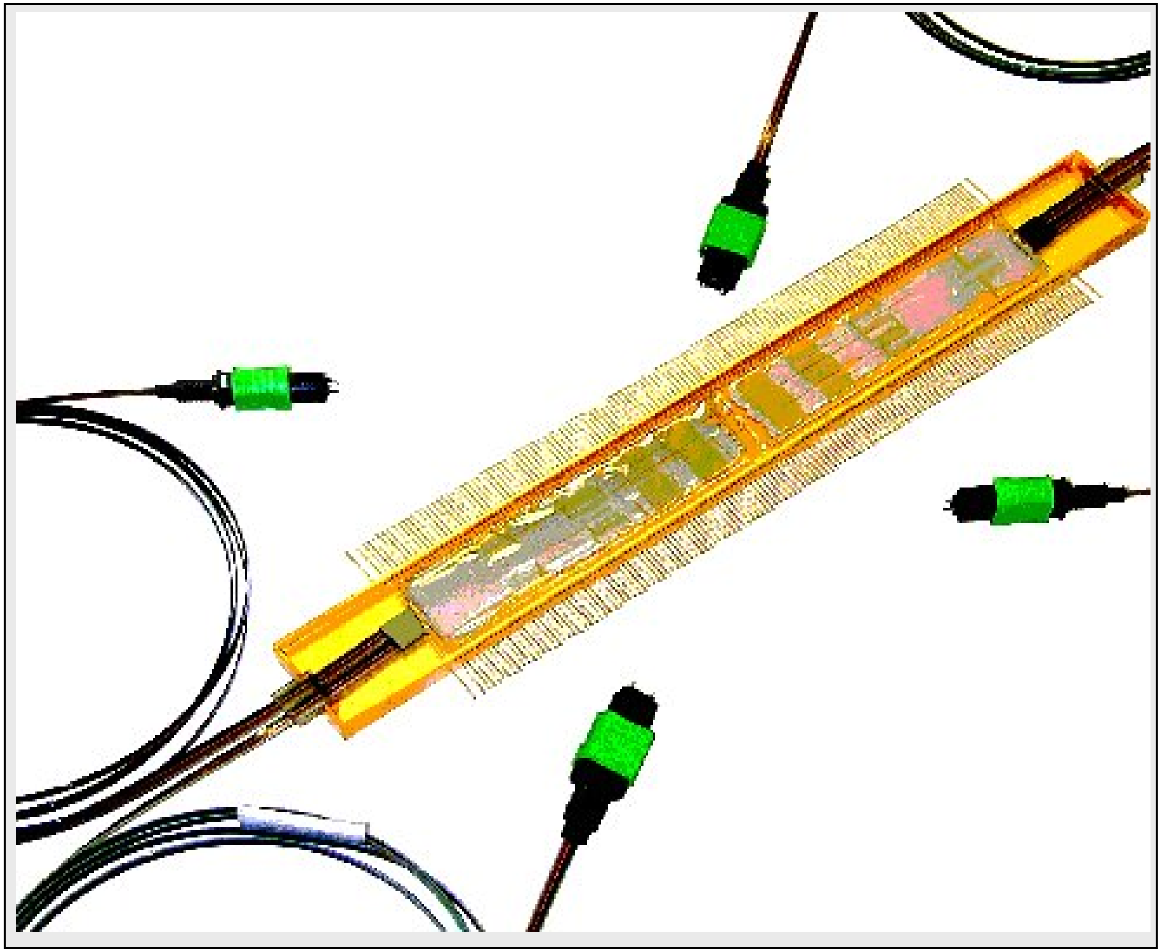
Properties of DOS switch:

+ polarization independence, step-like shape of the switching characteristics;

- high switching voltage (± 60 V)

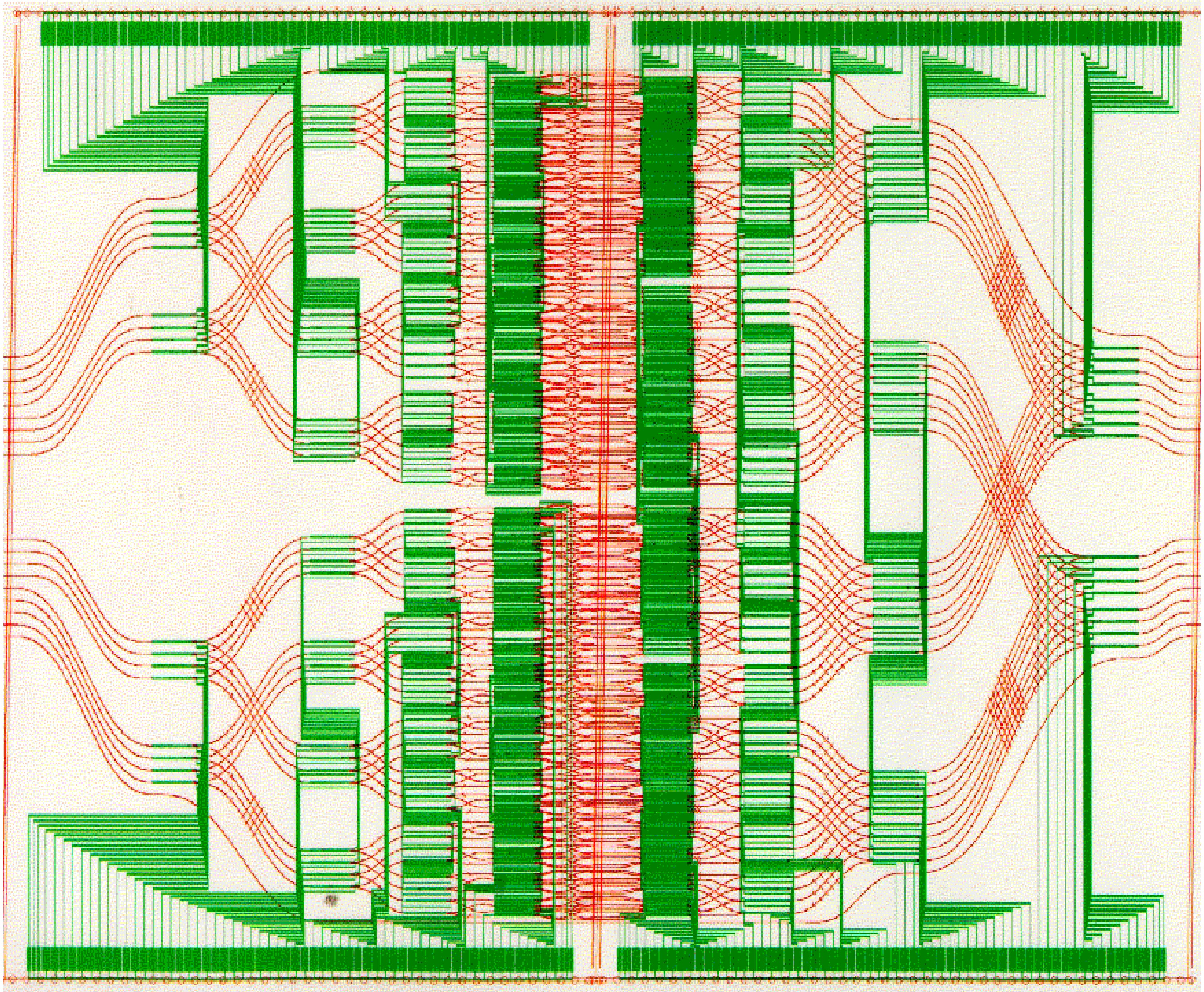
„Space“ 16×16 switching matrix in Ti:$LiNbO_3$ (2×20×5 mm)

„Non-blocking“ architecture, 480 DOS switches. U= ± 45 V, IL < 15 dB,
$\tau \cong$ 5 ns, PMD < 1 ps, PMD compensation by quartz $\lambda/4$ plate



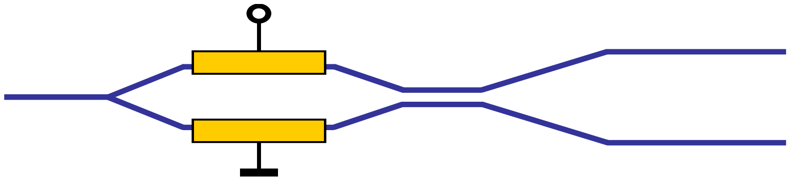
(Lucent, 2000)

Layout of optical waveguides and of electrode structure of the switching matrix



Examples of electro-optic and acousto-optic devices

Linear modulator for cable TV: MZ modulator + directional coupler



GENERAL SPECIFICATIONS

Material	$LiNbO_3$
Crystal orientation	x-cut, y-propagating
Electrical connectors (package)	SMA connectors
Operating wavelength	1535 - 1550nm
Fiber Options (1 meter fiber pigtails)	1. Fujikura SM 15-P-8/125-UV/UV-400 2. 3M FS-PM-7621 3. Corning SMF 28 4. Custom Fiber[2] (Customer supplied)

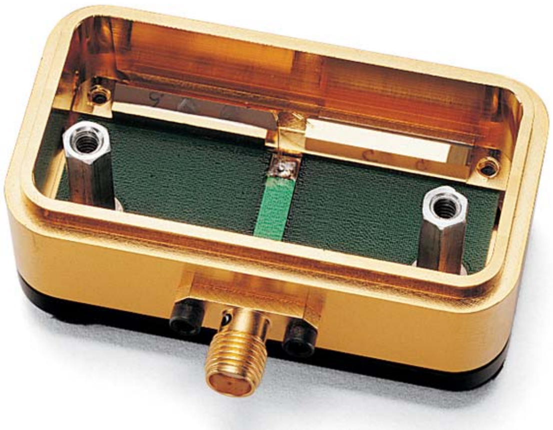
ABSOLUTE SPECIFICATIONS

Input optical power	200 mW maximum
Operating temperature	-25°C minimum, 75°C maximum
Storage temperature	-45°C minimum, 90°C maximum
Bias Port	
Applied DC Voltage	± 15 V maximum
RF Port	
Applied DC Voltage	0 V maximum
Applied RF Power	+ 27 dBm maximum

Overview of commerial acousto-optic and electro-optic devices

Electro-optic amplitude modulators (Newport)

	4102NF	4104NF	4101NF	4103
Type (1)	Broadband AM	Broadband AM	Resonant AM	Resonant AM
Operating Frequency	DC-200 MHz	DC-200 MHz	0.01-250 MHz	0.01-250 MHz
Wavelength Range	500-900 nm	900-1600 nm	500-900 nm	900-1600 nm
Material	$LiNbO_3$	$LiNbO_3$	$LiNbO_3$	$LiNbO_3$
Maximum $V\pi$ (2)	160 V @ 532 nm	300 V @ 1000 nm	16 V @ 532 nm	30 V @ 1000 nm
On:Off Extinction Ratio (3)	50:1	50:1	50:1	50:1
Maximum Optical Intensity (4)	0.5 W/mm^2 @ 532 nm	1 W/mm^2 @ 1300 nm	0.5 W/mm^2 @ 532 nm	1 W/mm^2 @ 1300 nm
Aperture Diameter	2 mm	2 mm	2 mm	2 mm
Insertion Loss (5)	<0.3 dB	<0.3 dB	<0.3 dB	<0.3 dB
RF Bandwidth	200 MHz	200 MHz	2-4% freq.	2-4% freq.
RF Connector	SMA	SMA	SMA	SMA
Input Impedance	10 pF	10 pF	50 Ω	50 Ω
Maximum RF Power	10 W	10 W	1 W	1 W
VSWR	NA	NA	<1.5	<1.5



Electro-optic phase modulators (Newport)

	4006	4002	4004	4005	4001NF	4003NF
Type (1)	Broadband	Broadband	Broadband	Resonant	Resonant	Resonant
Operating Frequency	DC-100	DC-100 MHz	DC-100 MHz	0.01-250	0.01-250 MHz	0.01-250 MHz
Wavelength Range	360-500 nm	500-900 nm	900-1600 nm	360-500 nm	500-900 nm	900-1600 nm
Material	$MgO:LiNbO_3$	$MgO:LiNbO_3$	$MgO:LiNbO_3$	$MgO:LiNbO_3$	$MgO:LiNbO_3$	$MgO:LiNbO_3$
Modulation Depth	40 mrad/V @ 364 nm	30 mrad/V @ 532 nm	15 mrad/V @1000 nm	0.27 - 0.8 rad/V @ 364 nm	0.2 - 0.6 rad/V @ 532 nm	0.1 - 0.3 rad/V @ 1000 nm
Maximum $V\pi$ (2)	79 V @ 364 nm	105 V @ 532 nm	210 V @ 1000 nm	3.8 - 11.7 V @ 364 nm	5 - 16 V @ 532 nm	10 - 31 V @ 1000 nm
Maximum Optical Intensity (4)	0.1 W/mm^2 @ 364 nm	2 W/mm^2 @ 532 nm	4 W/mm^2 @ 1064 nm	0.1 W/mm^2 @ 364 nm	2 W/mm^2 @ 532 nm	4 W/mm^2 @ 1064 nm
Aperture Diameter	2 mm	2 mm	2 mm	2 mm	2 mm	2 mm
RF Bandwidth	100 MHz	100 MHz	100 MHz	2-4% freq.	2-4% freq.	2-4% freq.
RF Connector	SMA	SMA	SMA	SMA	SMA	SMA
Input Impedance	20 pF	20 pF	20 pF	50 Ω	50 Ω	50 Ω
Maximum RF Power	10	10 W	10 W	1	1 W	1 W
VSWR	NA	NA	NA	NA	<1.5	<1.5

Electro-optic modulators

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- **1147 SERIES** RTP ELECTRO-OPTIC MODULATORS AND Q-SWITCHES

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- **1150 SERIES** BBO POCKELS CELL Q-SWITCHES

Transverse Field KD*P Modulator

- **3079 SERIES** LOW VOLTAGE LIGHT MODULATORS

Isomet Corporation, Springfield, USA – traditional producer of acousto-optic devices

http://www.isomet.com

AO modulators for UV and VIS spectral domains

Model	Operating Wavelength Range	Crystal Material	Active Aperture (mm)	Typical Risetime (ns)	Modulation Bandwidth (MHz)	Center Freq. (MHz)
M1134-FS80L	UV	Fused Silica	3	55	10	80
1211-5-UV	UV	Quartz	5	113	5	110
M1088-FS110L	UV	Fused Silica	3	55	10	110
1211-UV	UV	Quartz	2	57	20	150
1212-2-949	UV	Quartz	2	25	20	150
1212	UV	Quartz	1	10	30	175
1212-248	UV	Quartz	1	10	30	200

1201E-1	VIS	Glass	1.7	46	7	40
1201E-964	VIS	Glass	3	93	10	70
OAM1060	VIS	TeO2 (S)	2	1000	0.2	80
M1115-FS80L-3	VIS	Fused Silica	3(H)x14(W)	170	10	80
1205C-x	VIS	PbMo04	1 / 2 / 3	25	15	80
M1133-aQ80L	VIS	Quartz	1.5 / 2	114	10	80
OAM1020	VIS	TeO2 (S)	3	1000	0.2	110
1211	VIS	Quartz	2	57	10	110
1211-3-985	VIS	Quartz	2.7	57	20	110
1206C	VIS	PbMo04	1	15	25	110
1206C-833	NUV, VIS	TeO2	1	15	25	110
1206C-2-1002	NUV, VIS	TeO2	2	30	25	110
1250C-829A	NUV, VIS	TeO2	0.45	9	50	260
1250C	VIS	PbMo04	0.75	10	50	200
1250C-848	VIS	TeO2	0.5	7	50	200
1250C-974	VIS	TeO2	0.4	7	50	200
M1067-T200L	VIS	TeO2	0.2	7	50	200
1260-1044	VIS	TeO2	0.2	6	100	350

AO modulators for IR domain

Model	Operating Wavelength Range	Crystal Material	Active Aperture (mm)	Typical Risetime (ns)	Modulation Bandwidth (MHz)	Center Freq. (MHz)
1201E-2	NIR	Glass	1.7	93	3.8	40
1202-4	NIR	Glass	4(H)x14(W)	350	10	40
M1137-SF40L	NIR	Glass	1.5	191	10	40
1205C-x-NIR	NIR	PbMo04	1 / 2	25	15	80
1205C-1023	NIR	PbMo04	0.6	25	15	80
1205C-843	NIR	PbMo04	0.5	25	15	80
M1142-SF80L	NIR	Glass	0.5	40	15	80
M1080-T80L	NIR	TeO2	1.5	77	15	80
M1135-T80L	NIR	TeO2	3	245	15	80
1206C-NIR	NIR	PbMo04	1	15	25	110
1250C-868	NIR	TeO2	0.5	7	25	150
1250C-NIR	NIR	PbMo04	0.75	10	50	200

1207B-3	IR	Ge	3	70	8	40
1210	mid-IR	Ge	4	500	10	81 / 105
1208-6-4(M)	mid-IR	Ge	6(H)x14(W)	500	10	50
1207B-6	IR	Ge	6	700	10	40
1208-6-955M	IR	Ge	6(H)x14(W)	700	10	40
1209-7-993M	IR	Ge	7(H)x14(W)	830	10	40
1209-7-1064M	IR	Ge	7(H)x14(W)	830	10	40
1209-7-1112M	IR	Ge	7(H)x14(W)	830	10	40
1209-9-1010M	IR	Ge	9(H)x20(W)	830	2.5	40
AOM6x0-H	IR	Ge	7(H)x30(W)	830	10	40 / 50

Multichannel AO modulators

Model	Channels	Spectral Range (µm)	Material	Active Aperture (mm)	Typical Risetime (ns)	Information Bandwidth (MHz)	Center Freq. (MHz)
M1140	4	0.45-0.67	PbMoO4	0.7	25	15	110
8080	8	0.45-0.67	PbMoO4	0.7	36	9	80
M8080C	8	0.488-0.633	PbMoO4	0.5	55	6	80
M9080C	8,collinear	0.45-0.67	PbMoO4	0.7	36	9	90
G7060	6	2.5-11.0	Ge	0.8	70	5	70

AO deflectors

Model	Operating Wavelengths	Material	Resolution	Time Aperture (us)	Sweep Bandwidth (MHz)	Center Freq. (MHz)
1211-5BS-1045	UV	Quartz	35	0.87	40	110
D1155-T75S	405nm	TeO2 (S)	140	14.5	10	75

1205C-2	VIS	PbMoO4	16	0.55	30	80
LS55-V	VIS	TeO2 (S)	450	11.3	40	80
LS110-VIS	VIS	TeO2 (S)	1100	22.7	50	100
LS110A-VIS-XY	VIS	TeO2 (S)	750x750	15	50	100
OAD948	488nm	TeO2 (S)	600	12.3	50	100
OAD1020	532nm	TeO2 (S)	600	12.3	50	100
1206C-1002	NUV, VIS	TeO2	35	0.7	50	110
OPP834	VIS	PbMoO4	520	5.2	100	200
1250C-BS-960A	VIS	PbMoO4	192	1.6	120	190

OAD1550-XY	1550nm	TeO2 (S)	200x200	10	20	40
LS110-NIR	NIR	TeO2 (S)	1100	22.7	25	50
LS110A-NIR-XY	NIR	TeO2 (S)	375x375	15	25	50
1205C-x-804B	NIR	PbMoO4	66	1.6	40	80
OAD1121-XY	810nm	TeO2 (S)	500x500	13	40	80
LS55-NIR	NIR	TeO2 (S)	450	11.3	40	80
D1135-T110L	NIR	TeO2	35	0.7	50	110
1250-BS-926	NIR	PbMoO4	70	1	70	145
1250C-BS-943A	NIR	PbMoO4	190	1.6	120	185

1208-6BS-955M	IR	Ge	50	2.5	20	40
1209-7BS-986	IR	Ge	50	2.5	20	40
AOM6x0-H	IR	Ge	100	5.5	20	40 / 50
LS50XY	IR	Ge	50x50	1.27	40	70
LS600-1011	IR	Ge	436	10.9	40	70
LS600-4	IR	Ge	545	13.6	40	70

AO frequency shifters

Model	Operating Wavelengths	Material	Active Aperture (mm)	Center Frequency (MHz)	Frequency Range (MHz)
OAM1059-V31	633nm	TeO2 (S)	1.5	10	+/- 0.5
OAM1059A	633nm	TeO2 (S)	1.5	15	+/- 1.0
1201E-1	VIS	Glass	1.7	40	+/- 7.0
1201E-2	NIR	Glass	1.7	40	+/- 7.0
OAM1141-T40-2	633nm	TeO2 (S)	2	40	+/- 1.0

OAM1141-T80-2	633nm	TeO2 (S)	2	80	+/- 1.0
1205-1054	VIS	PbMo04	1	80	+/- 5
1205-1069	VIS	PbMo04	1	160	+/- 5
M1141-P80-1	VIS	PbMo04	1	80	+/- 5
1205-1118	VIS	PbMo04	2	80	+/- 5
1205C-1-869	VIS,NIR	PbMo04	1 / 2	80	+/- 20

1206C	VIS,NIR	PbMo04	1	110	+/- 25
1250C	VIS,NIR	PbMo04	0.75	200	+/- 50
1250C-829A	NUV,VIS	TeO2	0.45	260	+/- 50
OPP-1	VIS	PbMoO4	1.5	300	+/- 100

1210	mid-IR	Ge	4	81 / 105	+/- 10
1207B-6	IR	Ge	6	40	+/- 10
1207B-3-80	IR	Ge	3	80	+/- 2.5

AO Q-switches

Model	Cooling	Centre Frequency (MHz)	Material	Active Aperture (mm)	Max RF Power (W)	Damage Threshold (MW/cm2)
Q1072-SF24L	Conduction	24	SF10	1.5	5	>300
Q1058C-SFxxL-H	Conduction	24/27	SF10	1.0/1.5	5	>300
Q1025-TxxL-H	Conduction	27/80	TeO2	1.0	3	>250
Q1025-SFxxL-H	Conduction	41/80	SF10	1.0	3	>300
Q1080C-TxxL-H	Conduction	41/ 68 / 80	TeO2	1.5	4	>250
Q1087-aQ80L	Conduction	80	Quartz	1.0	6	>500
Q1137-SFxxL-H	Conduction	41/80	SF57	1.0 / 1.5	6	>300
Q1162-SFxxL-H	Conduction	41/80	SF10	1.0	6	>300
-						
Q1119-aQxxL-H	Conduction	41/ 80	Quartz	1.0 / 1.5	10	>500
Q1119-FSxxL-H	Conduction	41/ 80	Fused Silica	1.0 / 1.5	10	>500
Q1133-aQxxL-H	Conduction	41/ 68 / 80	Quartz	1.0 to 2.0	10	>500
Q1133-FSxxL-H	Conduction	41/ 68 / 80	Fused Silica	1.0 / 1.5	10	>500
-						
Q1062-FSxxL-H	Water	24/ 27	Fused Silica	1.5 to 6.0	60	>500
Q1062-FSxxS-H	Water	24/ 27	F.Silica (Shear)	1.5 to 5.5	60	>500
Q1083-FSxxL-H	Water	24/ 27 / 41	Fused Silica	1.5 to 6.0	60	>500
Q1083-FSxxS-H	Water	24/ 27 / 41	F.Silica (Shear)	1.5 to 5.5	60	>500

AO tuneable filters

Model	Spectral Range (μm)	Active Aperture (sq. mm)	Acceptance Angle (Deg.)	Optical Bandwidth (nm)	Drive Frequency (MHz)
AOLF-615-1049	VIS	2.5x2.5	3.5 - 4.5	1.0 - 6.0	109 - 65
AOLF-615-1082	VIS	2.5x2.5	3.5 - 4.5	1.0 - 6.0	109 - 65
AOTF614-08	VIS,NIR	5x5	3.5 - 6.0	1.0 - 22.0	140 - 35
AOTF614-16	VIS,NIR	5x5	2.5 - 4.2	0.6 - 11.0	140 - 35
AOTF614-24	VIS,NIR	5x5	3.5 - 6.0	0.4 - 7.0	140 - 35
AOTF920-14	NIR	5x5	3.4 - 6.1	2.0 - 27.0	95 - 26
AOTF920-20	NIR	5x5	2.6 - 4.9	1.5 - 18.5	95 - 26
AOTF920-24	NIR	5x5	2.8 - 5.0	1.0 - 15.5	95 - 26
AOTF1331	mid-IR	7x7	5	30 - 50	24 - 39
AOTF1550-SLS	1550nm	3x3	-	2	81 - 84
AOTF1110-VB	VIS,NIR	10x10	5.7 (nominal)	Variable	80 - 50
OSTF	VIS-NIR	5x5	4 (nominal)	1.0 - 12	110 - 45

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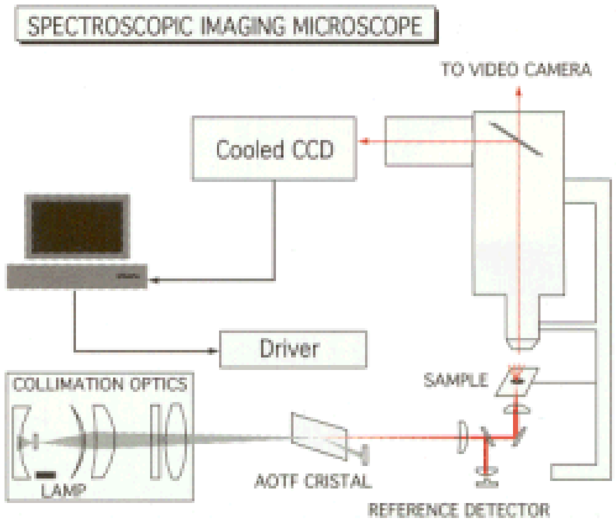
AO tuneable filters

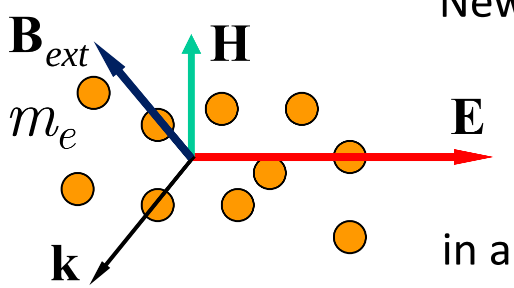
AOTF MODEL	OPTICAL RANGE (nm)	LIGHT SOURCE [1/N]:Laser /lines [2]:Lamp	OPTICAL TRANS. (%)	APERTURE (mm^2)	FIELD OF VIEW	AO EFF. (%) linear pol.	SPECTRAL RES. (nm) -3dB	MAX RF POWER
AOTFnC-UV	350-430	[1]	70-90	2×2	1°	85	1-2	2
AOTF-1	360-530	[2]	70-90	2×2	1.5°	85	1.5-5	0.2×N
AOTF-2	360-530	[1/4] or [2]	80-90	2×2	1.5°	85	1.5-5	0.2×N
AOTF-3	400-700	[2]	>90	5×5	5°	80	5-30	2
AOTF-5	480-620	[2]	>95	5×5	8°	80	3-10	2
AOTF-6	500-850	[1/1] or [2]	>95	5×5	3°	80-60	1-3	1
AOTF-7	600-900	[1/1] or [2]	>95	5×5	4°	70	<4	1.5
AOTF-7A	600-900	[1/1] or [2]	>95	10×10	4°	70	7-10	2
AOTF-8	800-1800	[1/1] or [2]	>95	5×5	4°	60	2-15	2



Fundamentals of magneto-optics

Drude model of a magneto-optic medium

„Free“ electron gas in an electromagnetic field in presence of a static external magnetic field



Newton’s equation of motion for a charge *q* with an effective mass *m*:

$$-m\ddot{\mathbf{r}} - m\gamma\dot{\mathbf{r}} + q\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_{ext}\right) = \mathbf{0}$$

For time-harmonic elmg. field $\mathbf{E} = \mathbf{E}_0 \exp\left(-i\omega t\right)$

in a steady state, $\mathbf{r} = \mathbf{r}_0 \exp(-i\omega t)$, it holds:

$$m\omega^2\mathbf{r}_0 + im\gamma\omega\mathbf{r}_0 + q\left(\mathbf{E}_0 - i\omega\mathbf{r}_0 \times \mathbf{B}_{ext}\right) = \mathbf{0}.$$

Let us choose the direction of an external DC magnetic field as *z*-axis, $\mathbf{B}_{ext} = B_{ext}\mathbf{z}^0$.

The equation then sounds

$$\begin{pmatrix} m(\omega^2 + i\gamma\omega) & i\omega q B_{ext} & 0 \\ -i\omega q B_{ext} & m(\omega^2 + i\gamma\omega) & 0 \\ 0 & 0 & m(\omega^2 + i\gamma\omega) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = -q \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

where x_0, y_0, z_0 are components of $\mathbf{r}_0$ and E_x, E_y, E_z are components of $\mathbf{E}_0$.

Susceptibility of a medium in a magnetic field

Minor formal modification gives

$$\begin{pmatrix} \omega^2 + i\gamma\omega & i\omega\omega_c & 0 \\ -i\omega\omega_c & \omega^2 + i\gamma\omega & 0 \\ 0 & 0 & \omega^2 + i\gamma\omega \end{pmatrix} \cdot \mathbf{r}_0 = \mathbf{M} \cdot \mathbf{r}_0 = -\frac{q}{m}\mathbf{E}_0,$$

where $\omega_c = \frac{qB_{ext}}{m}$ is a cyclotron frequency.

Obviously,

$$\mathbf{r}_0 = -\frac{q}{m}\mathbf{M}^{-1} \cdot \mathbf{E}_0,$$

$$\mathbf{P}_0 = qn\mathbf{r}_0 = -\frac{q^2 n}{m}\mathbf{M}^{-1} \cdot \mathbf{E}_0 = \varepsilon_0 \overline{\chi} \cdot \mathbf{E}_0,$$

$$\overline{\chi} = -\frac{q^2 n}{m\varepsilon_0}\mathbf{M}^{-1} = -\omega_p^2 \mathbf{M}^{-1}, \quad \omega_p = |q|\sqrt{\frac{n}{m\varepsilon_0}}$$ ... plasma frequency.

Explicit calculation of an inverse matrix and generalization to materials with a permittivity

$\lim\limits_{\omega \to \infty} \varepsilon(\omega) = \varepsilon_\infty$ gives $\boldsymbol{\varepsilon} = \varepsilon_\infty \left(\mathbf{I} + \overline{\chi}\right),$

Susceptibility of a medium in a magnetic field

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & i\varepsilon_{xy} & 0 \\ -i\varepsilon_{xy} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}, \qquad \varepsilon_{xx} = \varepsilon_\infty \left[1 - \frac{\omega_p^2 (\omega + i\gamma)}{\omega \left[(\omega + i\gamma)^2 - \omega_c^2 \right]} \right],$$

$$\varepsilon_{xy} = \varepsilon_\infty \frac{\omega_p^2 \omega_c}{\omega \left[(\omega + i\gamma)^2 - \omega_c^2 \right]}, \quad \varepsilon_{zz} = \varepsilon_\infty \left[1 - \frac{\omega_p^2}{\omega (\omega + i\gamma)} \right].$$

If the collision frequency $\gamma \ll \omega$ and $\omega_c < \omega, \;\; \varepsilon_{xx} \approx \varepsilon_{zz},$

the medium can be considered lossless, with *Hermitean permittivity,*

$\varepsilon_{xy} \simeq \dfrac{\varepsilon_\infty q^3 n}{\varepsilon_0 m^2 \omega^3} B_0$ is linearly dependent on the external magnetic field.

For the sake of simplicity, we will consider only such media.

Plane wave propagation in an isotropic magneto optic medium

"Rigorous" analysis based on the Fresnel dispersion equation
for a general anisotropic medium

$$\begin{pmatrix} \varepsilon_{xx} - l_y^2 - l_z^2 & i\varepsilon_{xy} + l_x l_y & l_x l_z \\ -i\varepsilon_{xy} + l_x l_y & \varepsilon_{xx} - l_x^2 - l_z^2 & l_y l_z \\ l_x l_z & l_y l_z & \varepsilon_{zz} - l_x^2 - l_y^2 \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The problem is rotationally symmetric with the axis given by the direction of the magnetic field; it is thus sufficient to analzye propagation in the *xz* plane, $l_y = 0$:

$$\begin{pmatrix} \varepsilon_{xx} - l_z^2 & i\varepsilon_{xy} & l_x l_z \\ -i\varepsilon_{xy} & \varepsilon_{xx} - l_x^2 - l_z^2 & 0 \\ l_x l_z & 0 & \varepsilon_{zz} - l_x^2 \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For real media and magnetic field $B_{ext} \leq 1$ T, $\omega_c \ll \omega$, $\varepsilon_{xx} \approx \varepsilon_{zz}$ and $\left|\varepsilon_{xy}\right| \ll \varepsilon_{xx}$.

Wave vector surfaces of an isotropic medium under external magnetic field

The dispersion equation (the determinant of the matrix) is then

$$\left[(\varepsilon_{xx} - l_x^2 - l_z^2)(\varepsilon_{xx} - l_z^2) - \varepsilon_{xy}^2\right](\varepsilon_{zz} - l_x^2) - (\varepsilon_{xx} - l_x^2 - l_z^2) l_x^2 l_z^2 = 0,$$

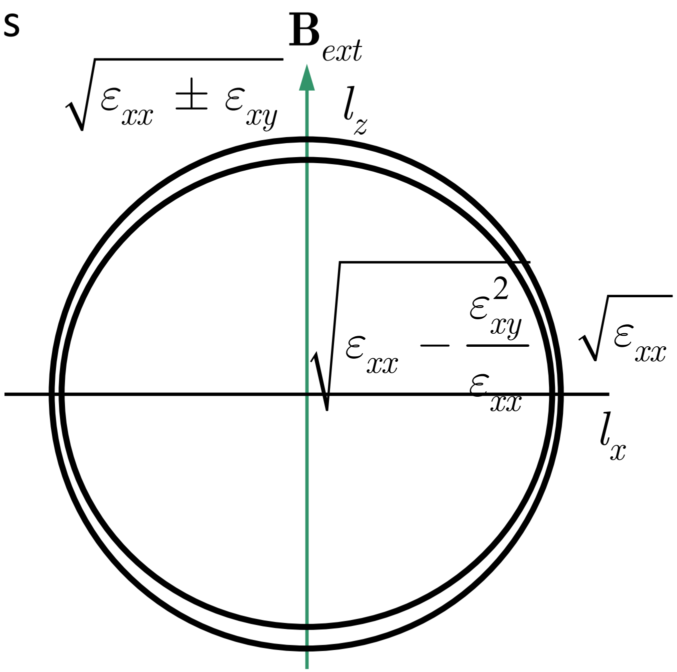
which for $\varepsilon_{xx} \approx \varepsilon_{zz}$ is reduced to

$$\varepsilon_{xx}(\varepsilon_{xx}^2 - l_x^2 - l_z^2)^2 - \varepsilon_{xy}^2(\varepsilon_{xx} - l_x^2) = 0.$$

This is a bi-quadratic equation for l_z with solutions

$$l_z = \pm\sqrt{\varepsilon_{xx} - l_x^2 \pm \varepsilon_{xy}\sqrt{\frac{\varepsilon_{xx} - l_x^2}{\varepsilon_{xx}}}}.$$

The wave vector surfaces are then close to ellipsoids of rotation (or even spheres).



Eigenwaves of the isotropic medium under external magnetic field

Propagation in the direction of the magnetic field, $l_x = 0,\ l_z^2 = \varepsilon_{xx} \pm \varepsilon_{xy}$

$$\begin{pmatrix} \varepsilon_{xx} - \varepsilon_{xx} \mp \varepsilon_{xy} & i\varepsilon_{xy} & 0 \\ -i\varepsilon_{xy} & \varepsilon_{xx} - \varepsilon_{xx} \mp \varepsilon_{xy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Solution gives $E_{0z} = 0,\ \ E_{0x} = \pm iE_{0y} \ \ \Rightarrow$ *circularly polarized waves*.

Chiral medium:

$$\begin{pmatrix} \varepsilon - l_z^2 & 2igl_z & 0 \\ -2igl_z & \varepsilon - l_z^2 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Magneto-optic medium:

$$\begin{pmatrix} \varepsilon_{xx} - l_z^2 & i\varepsilon_{xy} & 0 \\ -i\varepsilon_{xy} & \varepsilon_{xx} - l_z^2 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In both media the polarization is rotated, but the chiral medium is reciprocal while the ***magneto-optic medium is not reciprocal: in the magneto-optic medium polarization is rotated in the same direction for both forward and backward directions of propagation – Faraday effect.*** (***The only effect*** suitable for the construction of simple and efficient optical isolators.)

Eigenwaves of the isotropic medium under external magnetic field

Propagation perpendicularly to the direction of magnetic field,

$$\begin{pmatrix} \varepsilon_{xx} & i\varepsilon_{xy} & 0 \\ -i\varepsilon_{xy} & \varepsilon_{xx} - l_x^2 & 0 \\ 0 & 0 & \varepsilon_{zz} - l_x^2 \end{pmatrix} \cdot \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$l_z = 0,\ l_x^2 = \begin{cases} \varepsilon_{xx}, \\ \varepsilon_{xx} - \varepsilon_{xy}^2 / \varepsilon_{xx}. \end{cases}$$

Voigt (Cotton-Mouton) effect

The wave polarized in the direction of magnetic field propagates with $l_x = \sqrt{\varepsilon_{xx}}$

The wave polarized perpendicularly to the magnetic field propagates with

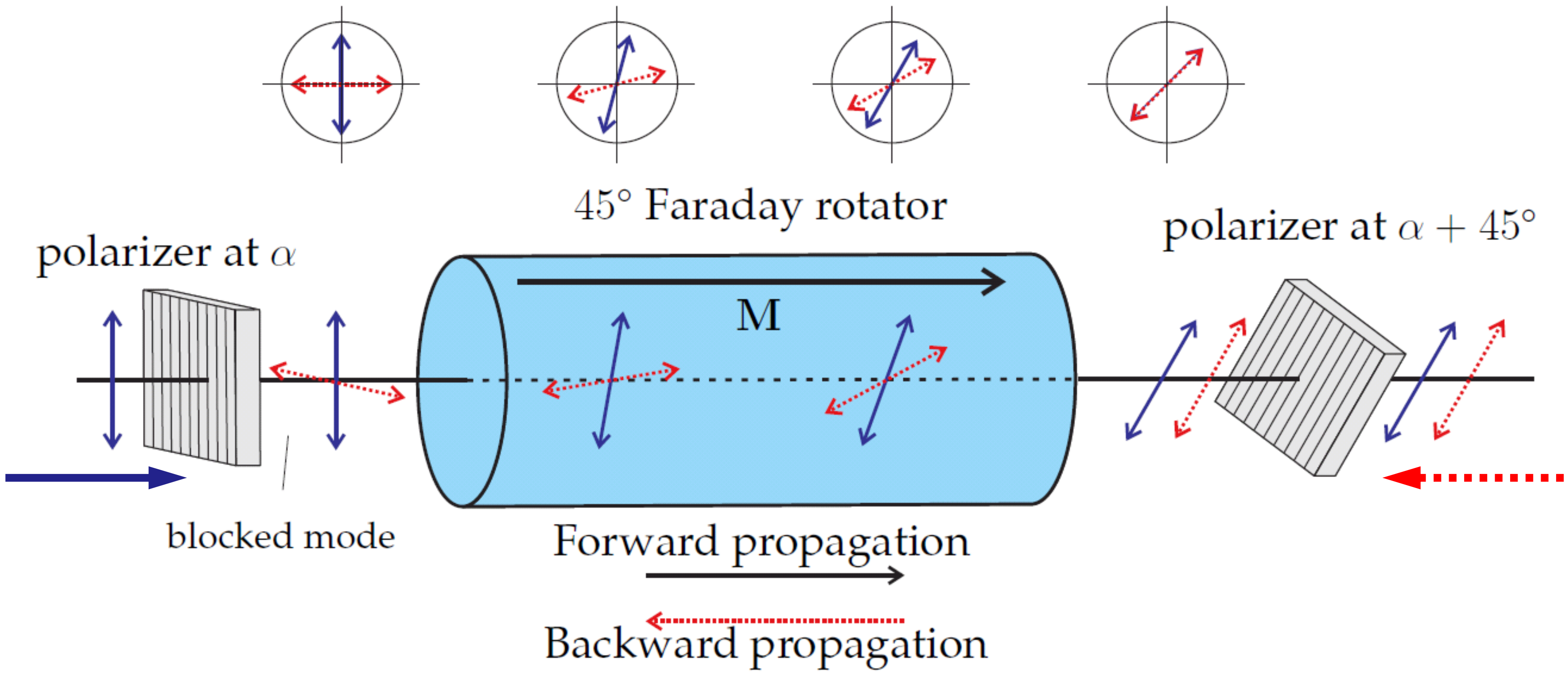
$$l_x = \sqrt{\varepsilon_{xx} - \frac{\varepsilon_{xy}^2}{\varepsilon_{xx}}} = \sqrt{\varepsilon_V},\ \varepsilon_V = \varepsilon_{xx} - \frac{\varepsilon_{xy}^2}{\varepsilon_{xx}} \ \ldots \text{ Voigt permittivity}$$

This wave has also a small but nonzero *longitudinal* component E_x,

$$\frac{E_x}{E_y} = -i\frac{\varepsilon_{xy}}{\varepsilon_{xx}},\quad \left|\frac{E_x}{E_y}\right| \ll 1.$$

The propagation constant of this wave, l_y, depends on **the square** of the magnetic induction, so that it is the same (up to the sign) for both directions of propagation.

The principle of the optic isolator

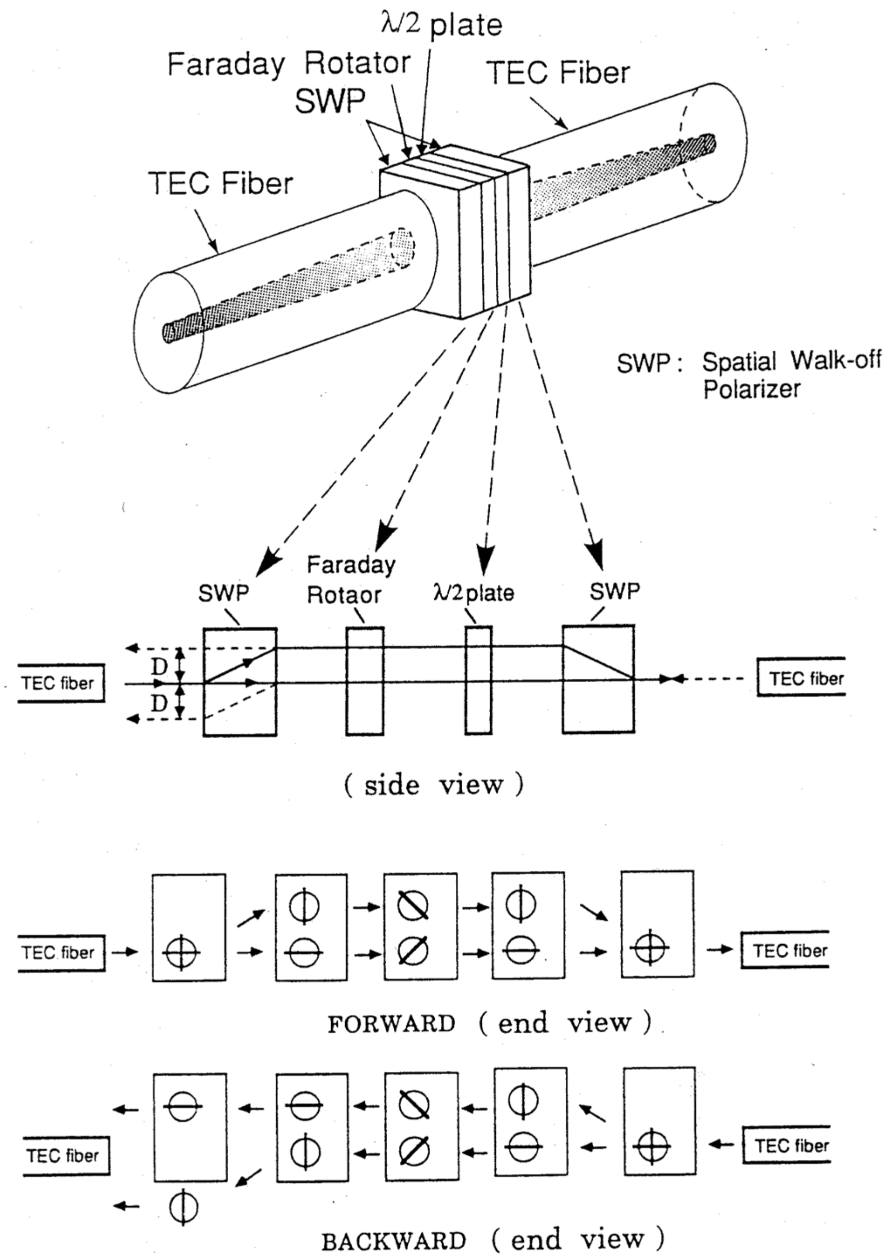


Only non-reciprocal effects can be used for the construction of true optical isolators!

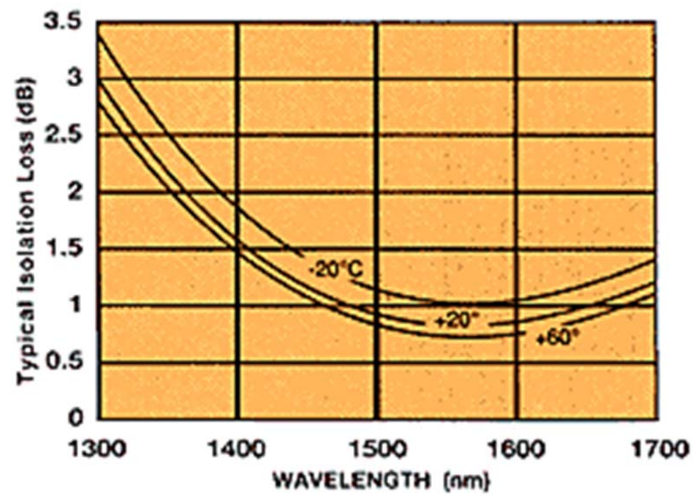
Paper of a fundamental importance on optical isolators:

D. Jalas, A. Petrov, M. Eich, W. Freude, S. Fan, Z. Yu, R. Baets, M. Popović, A. Melloni, J. D. Joannopoulos, M. Vanwolleghem, C. R. Doerr, and H. Renner, “What is–and what is not–an optical isolator,” Nature Photonics 7(8), 579–582 (2013).

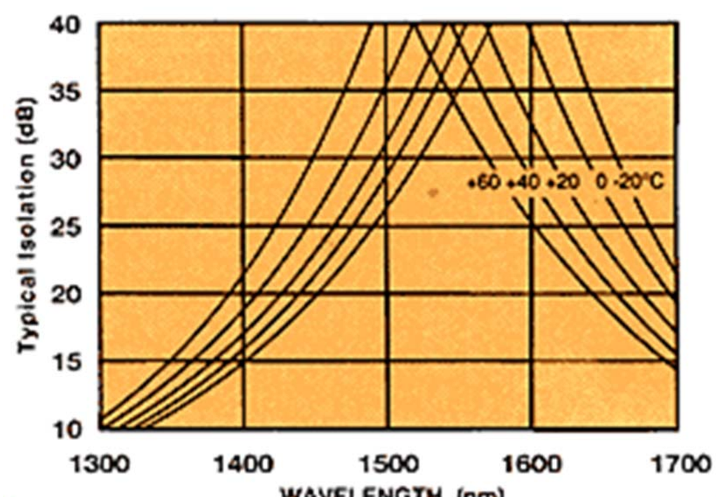
Fiber-optic polarization-independent optical isolator



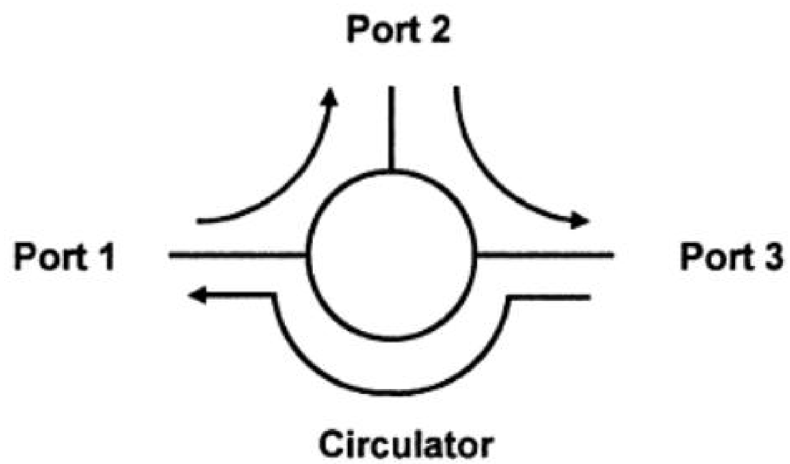
Insertion loss

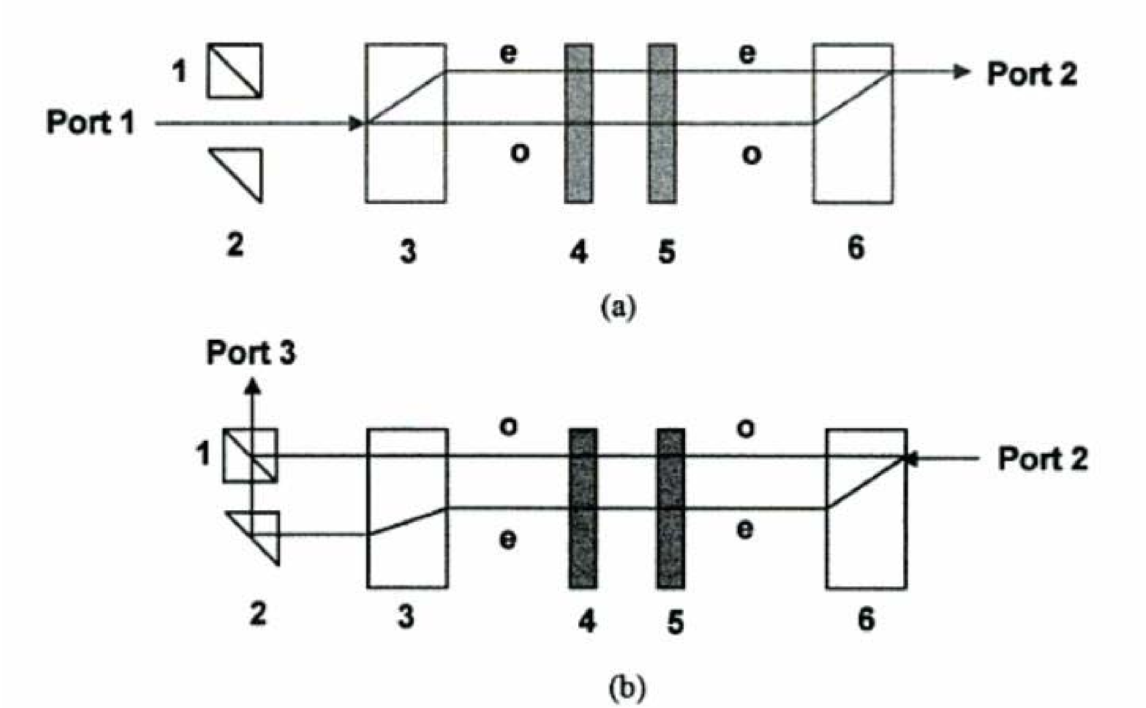


Isolation ratio



Fiber-optic polarization-independent optical circulator

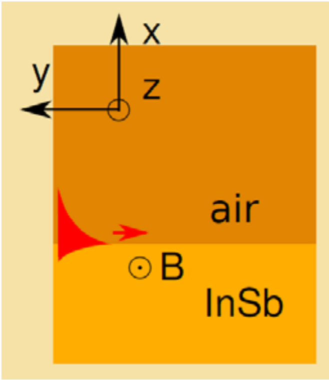


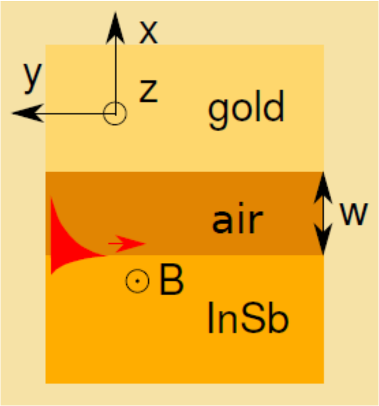


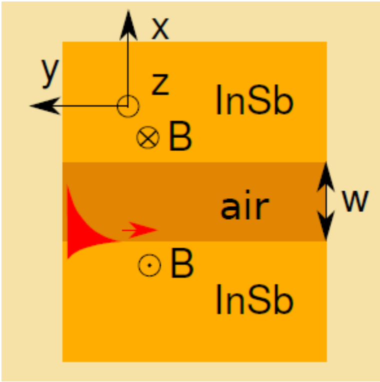
(1) beam splitting polarizer
(2) reflection prism
(3,6) birefringent crystals
(4) Faraday rotator
(5) half-wave plate

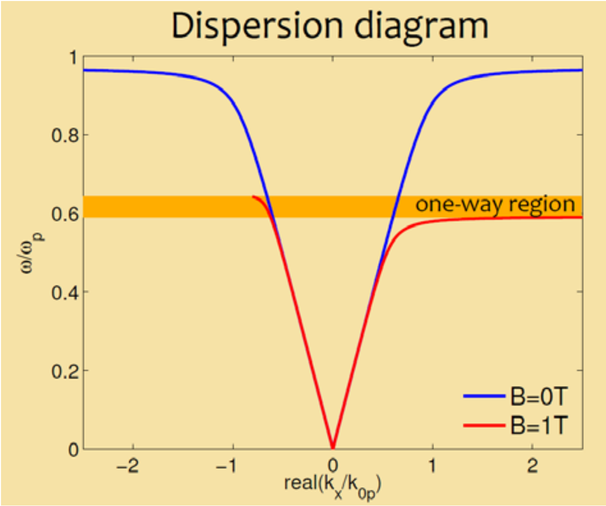
Waveguide isolators(?)

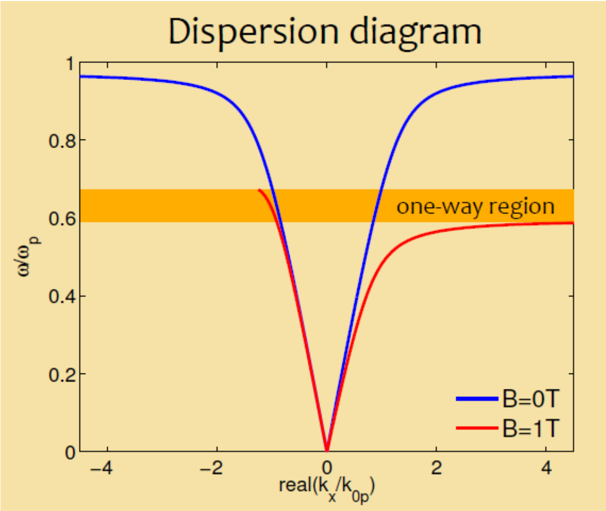
Faraday effect cannot be effectively utilized – short path, *polarization birefringence*.
Possible solution – transverse (Voigt) effect in an *asymmetric waveguide* (weak isolation)
Idea: MO splitting of dispersion characteristics of THz surface plasmons

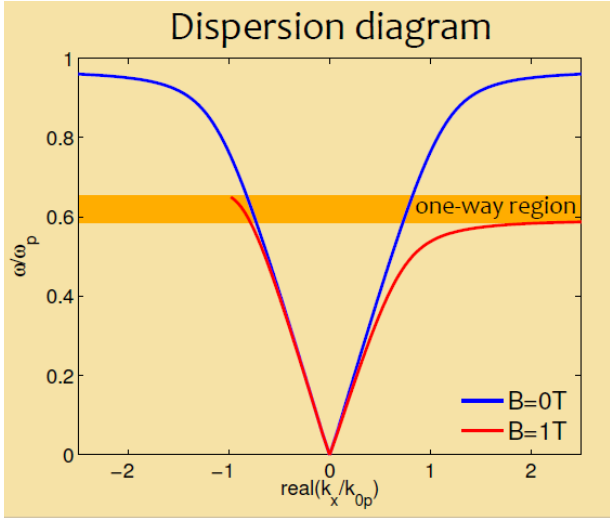






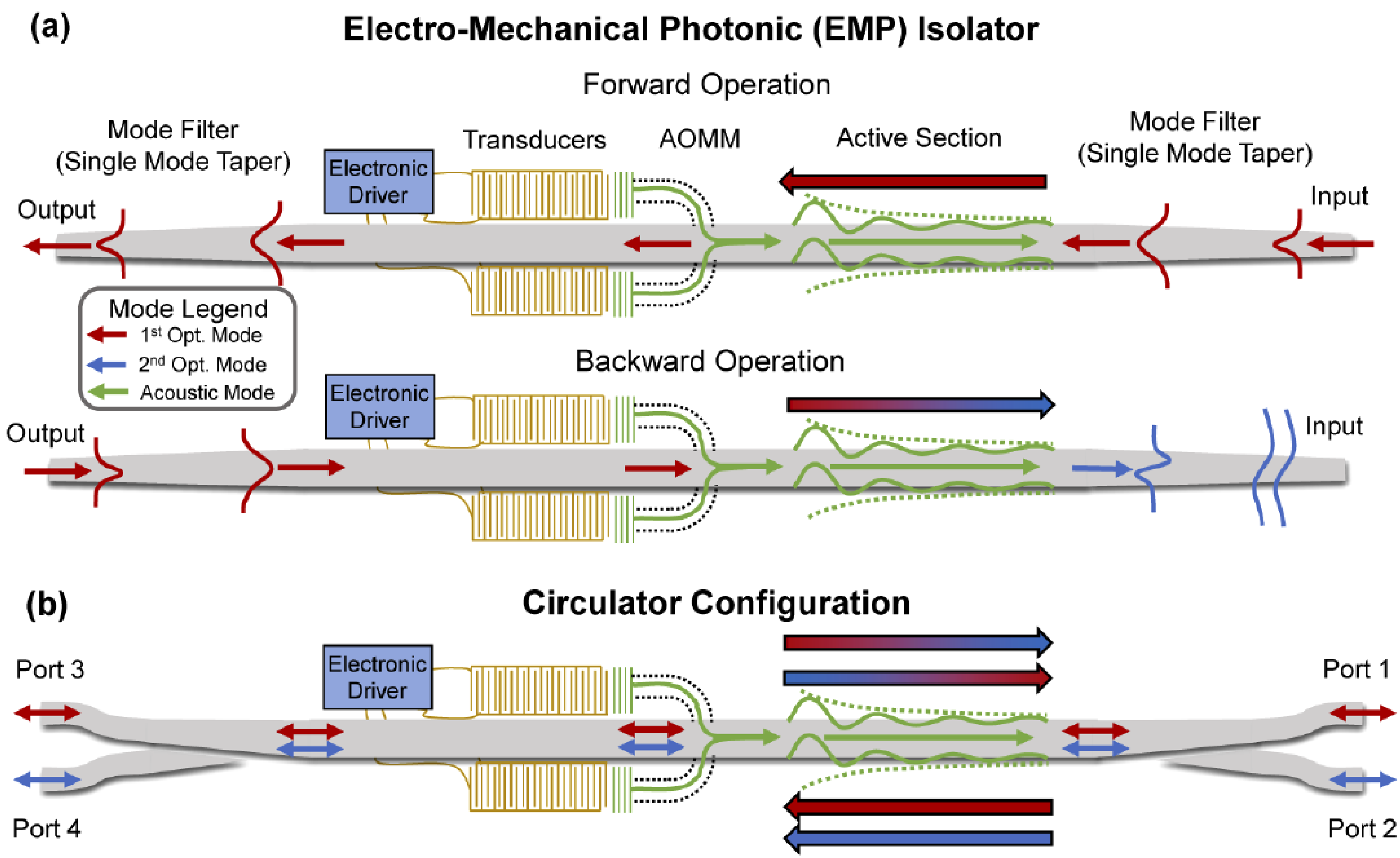




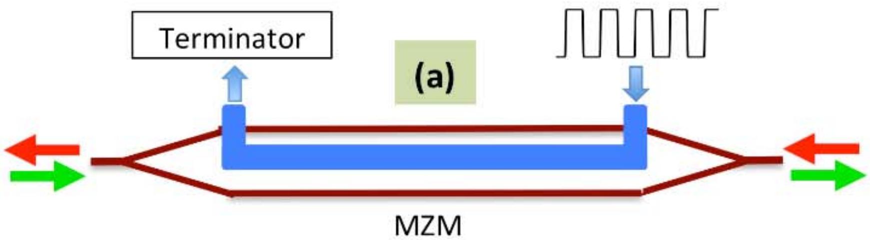


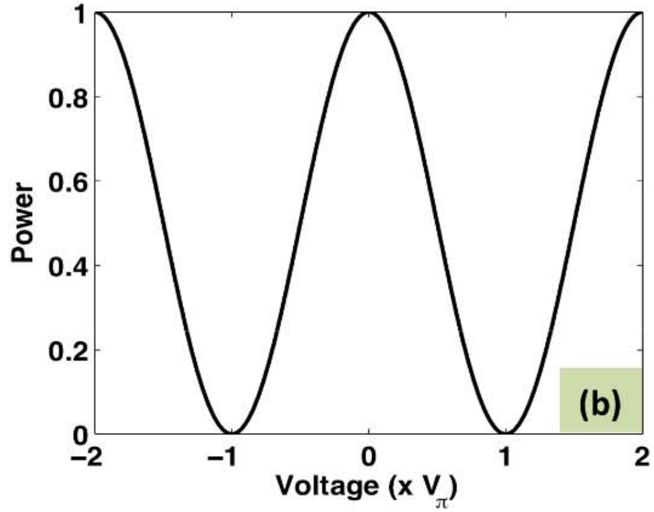
Optical isolators based on no-reciprocity due to time-dependent medium

Co-directional phase matching between backward propagating acoustic wave and two optical modes; acoustic frequency is f_a = 3 GHz

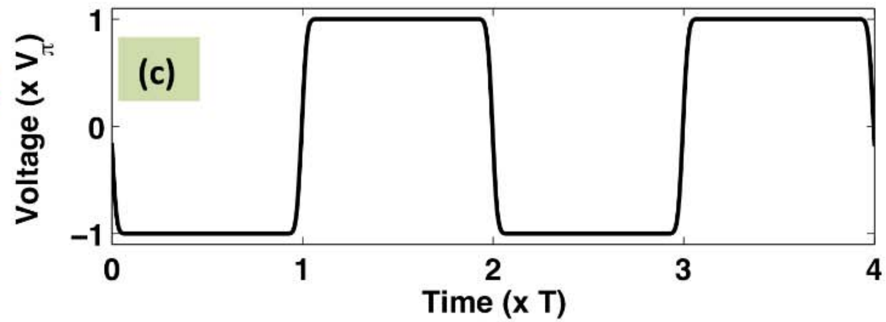


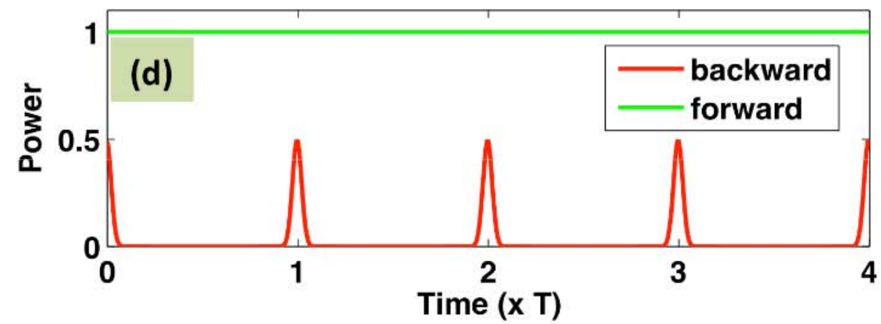
Travelling-wave Mach-Zehnder electro-optic modulator as an optical isolator

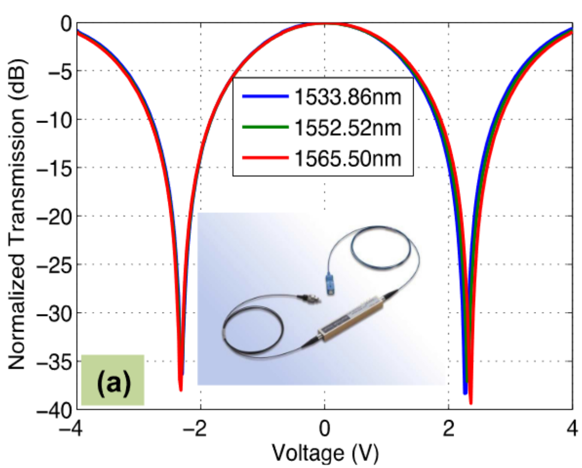


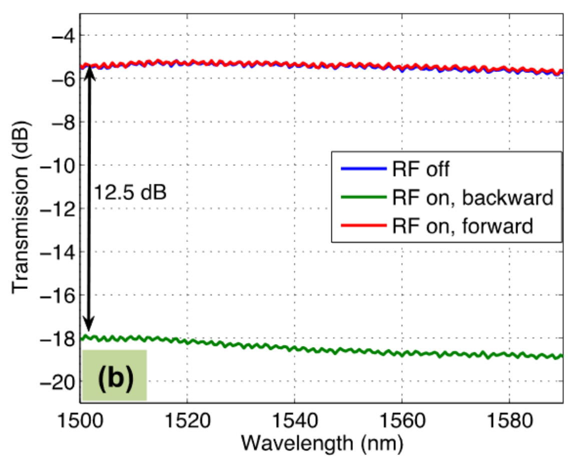


Square-wave $\pm U_\pi$ modulation at 2.75 GHz









Introduction to the theory of hyperbolic materials

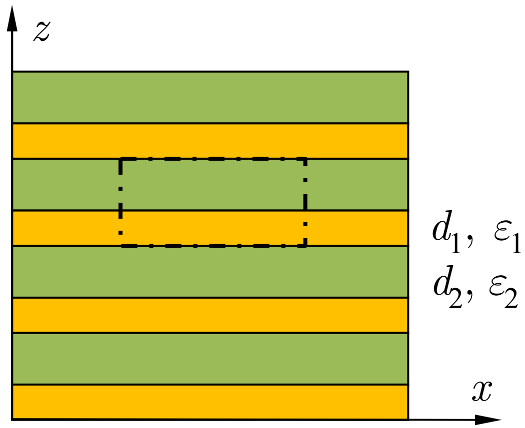
Elementary effective medium theory (EMT)

Let us consider a layered medium $\varepsilon_1,\ d_1$ and $\varepsilon_2,\ d_2$, $d_1,\ d_2 \ll \lambda$

For $\mathbf{E}$ parallel with interfaces, $\mathbf{E} = E_x \mathbf{x}^0,\ E_{x1} \approx E_{x2} = E_x$ due to continuity conditions.

The value of D_x averaged over the period $d_1 + d_2$ is $\bar{D}_x \approx \frac{\varepsilon_1 d_1 + \varepsilon_2 d_2}{d_1 + d_2} E_x = \varepsilon_{\|} E_x;$

let us define the filling factor $f = \frac{d_1}{d_1 + d_2}$; then $\varepsilon_{\|} = f\varepsilon_1 + (1-f)\varepsilon_2.$



For $\mathbf{E} = E_z \mathbf{z}^0,\ D_{z1} \approx D_{z2} = D_z$, while the averaged value of E_z is

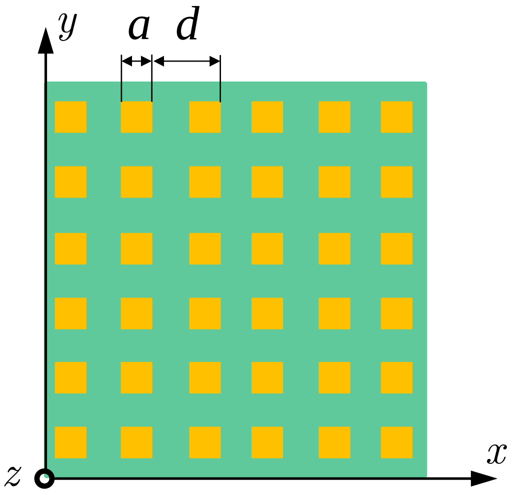
$$\bar{E}_z \approx \frac{E_{z1} d_1 + E_{z2} d_2}{d_1 + d_2} = \frac{d_1/\varepsilon_1 + d_2/\varepsilon_2}{d_1 + d_2} D_z \approx \frac{1}{\varepsilon_\perp} D_z;$$

thus $\frac{1}{\varepsilon_\perp} = f_1/\varepsilon_1 + (1-f_1)/\varepsilon_2$, $\varepsilon_\perp = \frac{\varepsilon_1 \varepsilon_2}{f_1 \varepsilon_2 + (1-f_1)\varepsilon_1}.$

Effective medium is thus uniaxially anisotropic, with the permittivity tensor

$$\bar{\varepsilon}_{eff} = \begin{pmatrix} \varepsilon_{\|} & 0 & 0 \\ 0 & \varepsilon_{\|} & 0 \\ 0 & 0 & \varepsilon_\perp \end{pmatrix}$$

J. C. Maxwell Garnett, "Colours in metal glasses and in metallic films,“ *Philosophical Transaction of the Royal Society London* **203**, 385-420 (1904),
S. Rytov, *J. Exp. Theor. Phys.*2, 466 (1956) (in Russian)



"Dual" ("nanowire") effective medium

Apparently, $\varepsilon_{\parallel} = f^2\varepsilon_1 + (1-f^2)\varepsilon_2$, $\quad \bar{\boldsymbol{\varepsilon}}_{eff} = \begin{pmatrix} \varepsilon_{\perp} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{\parallel} \end{pmatrix}$

$$f = \frac{a}{d}, \qquad \varepsilon_{\perp} \approx \frac{f\varepsilon_1\varepsilon_2}{f\varepsilon_2 + (1-f)\varepsilon_1} + (1-f)\varepsilon_2$$

Effective uniaxial anisotropic medium

Ideal (loss-less) metal-dielectric effective medium:

Example: Ag/SiO_2 @ $\lambda = 700$ nm: $\quad \varepsilon_m \doteq -22 + i0.67, \quad \varepsilon_d \doteq 2.12$

Layered medium

f	$\varepsilon_{\parallel}$	$\varepsilon_{\perp}$
0.2	-2.71	2.72
0.5	-9.94	4.69
0.7	-14.76	9.12
0.8	-17.18	17.25

Metal nanowire medium

f	$\varepsilon_{\parallel}$	$\varepsilon_{\perp}$
0.2	1.155	2.24
0.5	-3.91	3.41
0.7	-9.70	7.02
0.8	-13.317	14.23

Fresnel dispersion formula for a uniaxial medium

$$\Phi(\omega,\mathbf{l}) = \left(\varepsilon_{xx} - l_x^2 - l_y^2 - l_z^2\right)\left[\varepsilon_{xx}\varepsilon_{zz} - \varepsilon_{xx}\left(l_x^2 + l_y^2\right) - \varepsilon_{zz} l_z^2\right] = 0.$$

For a loss-less layered medium $\varepsilon_{xx} = \varepsilon_{\parallel} < 0, \ \varepsilon_{zz} = \varepsilon_{\perp} > 0.$

Thus

$$l_x^2 + l_y^2 + l_z^2 = \varepsilon_{\parallel}, \quad l = \pm i\sqrt{\left|\varepsilon_{\parallel}\right|}$$

"Ordinary" evanescent wave – *bulk plasmon* (non-propagating wave)

$$\frac{l_x^2 + l_y^2}{\varepsilon_{\perp}} + \frac{l_z^2}{\varepsilon_{\parallel}} = 1 \quad \text{or} \quad \frac{l_x^2 + l_y^2}{\varepsilon_{\perp}} = 1 + \frac{l_z^2}{\left|\varepsilon_{\parallel}\right|},$$

One-sheet hyperboloid of rotation

For a loss-less "nanowire" medium $\varepsilon_{xx} \approx \varepsilon_{\perp} > 0, \ \varepsilon_{zz} \approx \varepsilon_{\parallel} < 0.$

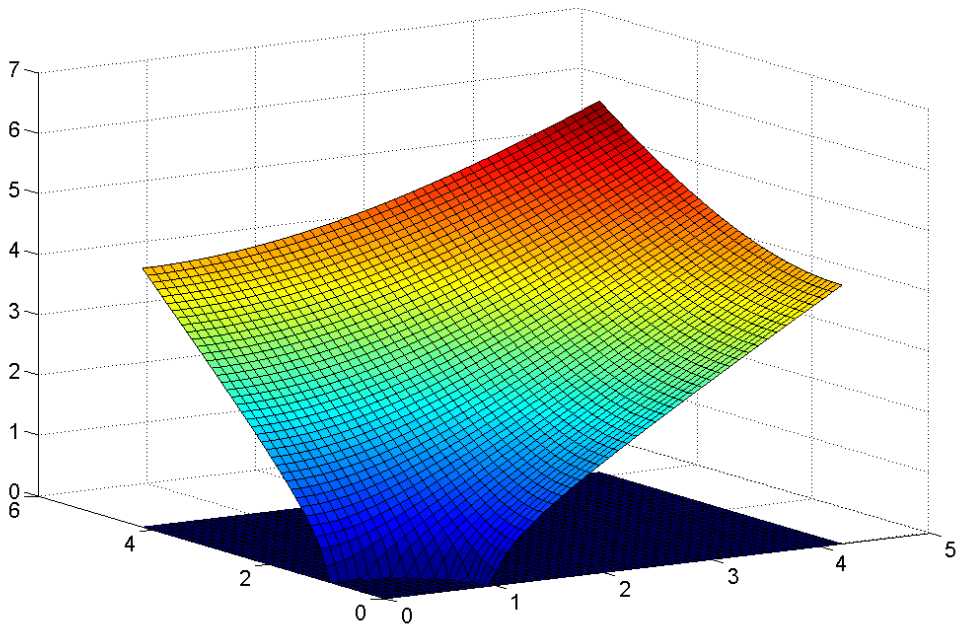
$$l_x^2 + l_y^2 + l_z^2 = \varepsilon_{\perp}, \quad l = \sqrt{\varepsilon_{\perp}}$$

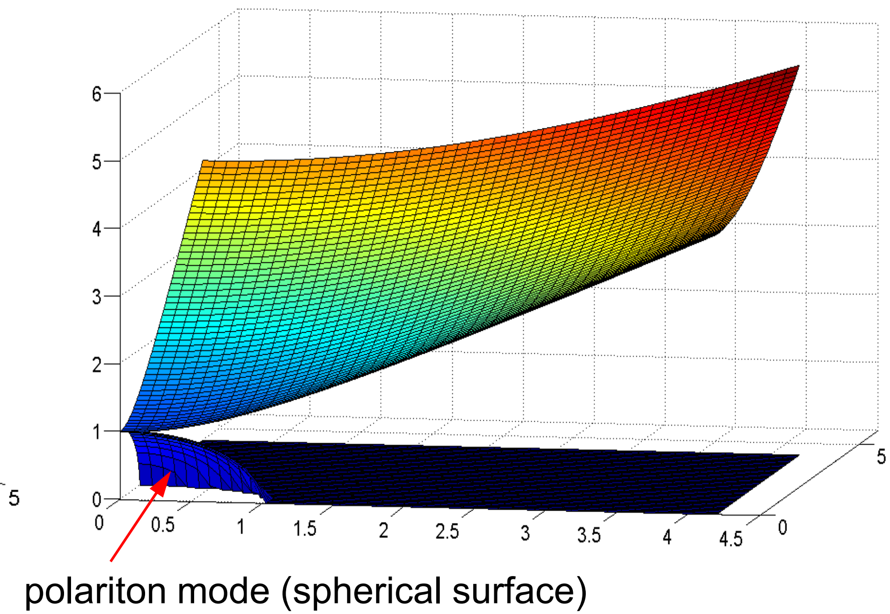
"Ordinary" propagating wave – *polariton mode*

$$\frac{l_x^2 + l_y^2}{\varepsilon_{\parallel}} + \frac{l_z^2}{\varepsilon_{\perp}} = 1 \quad \text{or} \quad \frac{l_x^2 + l_y^2}{\left|\varepsilon_{\parallel}\right|} = \frac{l_z^2}{\varepsilon_{\perp}} - 1,$$

Two-sheet hyperboloid of rotation (radius is positive for $\left|l_z\right| > \sqrt{\varepsilon_{\perp}}$).

Hyperbolic wave vector surfaces





One-sheet hyperboloid of rotation

$$\boldsymbol{\varepsilon}_{eff} = \begin{pmatrix} -1.1 & 0 & 0 \\ 0 & -1.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Two-sheet hyperboloid of rotation

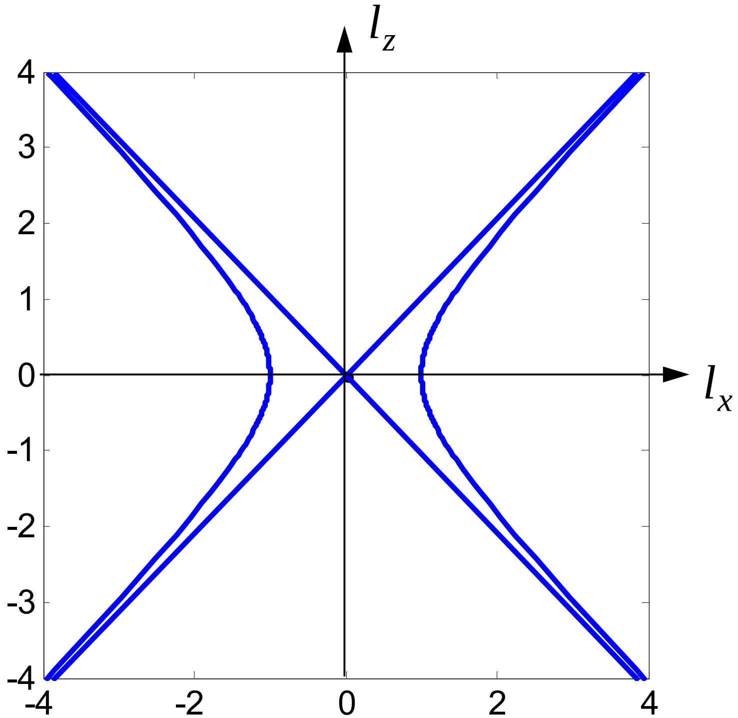
$$\boldsymbol{\varepsilon}_{eff} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.1 \end{pmatrix}$$

Hyperbolic wave vector surfaces – cut by *xz plane*

$$l_x^2 + l_z^2 = \varepsilon_{xx}, \quad \frac{l_x^2}{\varepsilon_{zz}} + \frac{l_z^2}{\varepsilon_{xx}} = 1;$$

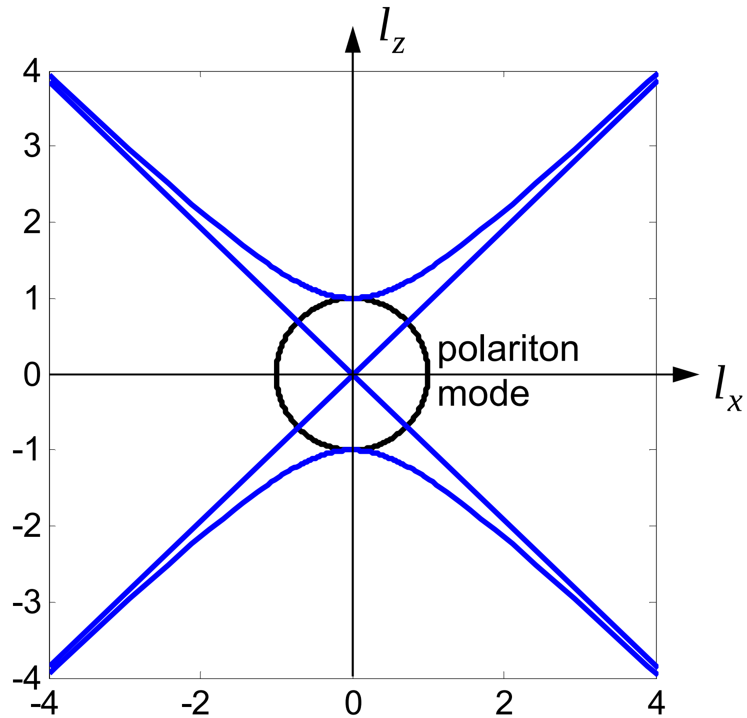
One-sheet hyperboloid of rotation

$$\bar{\boldsymbol{\varepsilon}}_{eff} = \begin{pmatrix} -1.1 & 0 & 0 \\ 0 & -1.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Two-sheet hyperboloid of rotation

$$\bar{\boldsymbol{\varepsilon}}_{eff} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.1 \end{pmatrix}$$



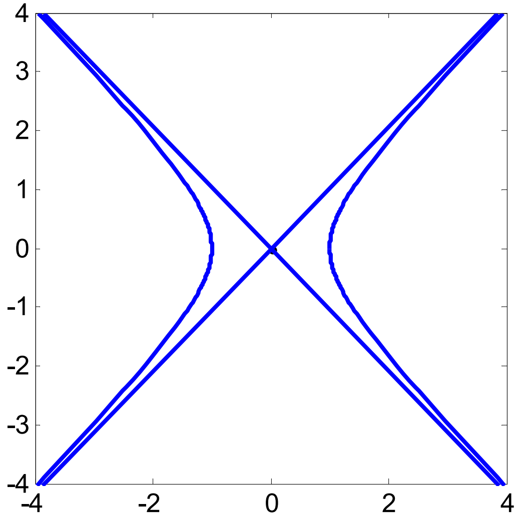
Hyperbolic surfaces of complex wave vectors

One-sheet hyperboloid of rotation

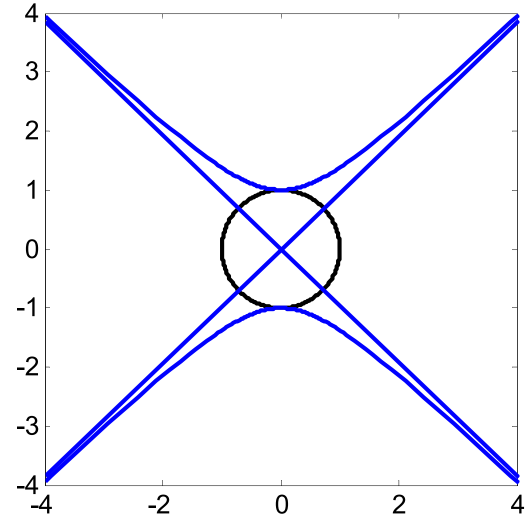
$$\bar{\varepsilon}_{eff} = \begin{pmatrix} -1.1 & 0 & 0 \\ 0 & -1.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

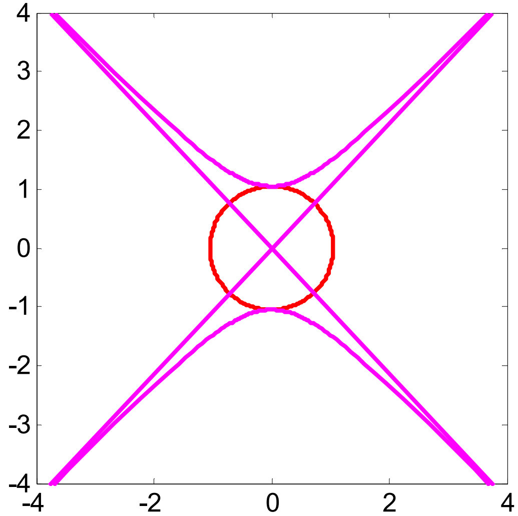
Two-sheet hyperboloid of rotation

$$\bar{\varepsilon}_{eff} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.1 \end{pmatrix}$$

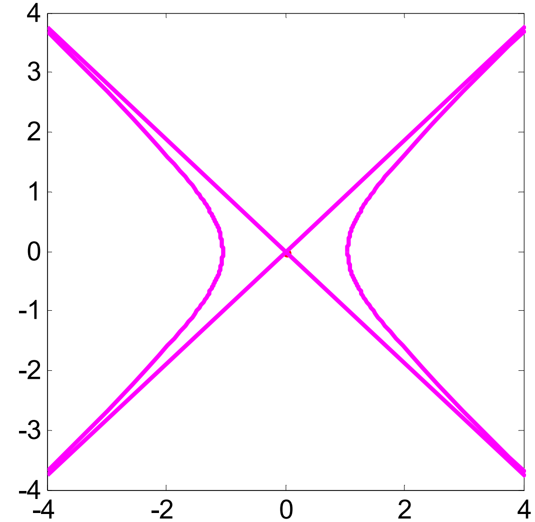


$\mathrm{Re}\{l_x\}, \ \mathrm{Re}\{l_z\}$





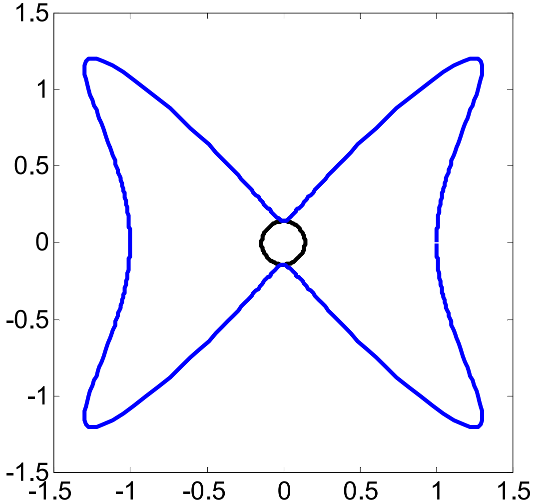
$\mathrm{Im}\{l_x\}, \ \mathrm{Im}\{l_z\}$



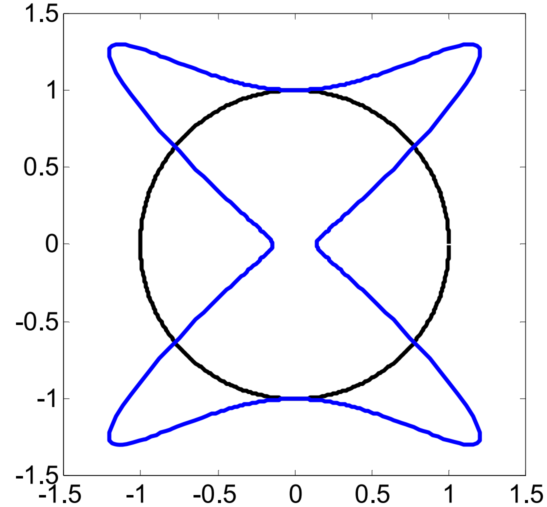
Complex wave vector surfaces in lossy hyperbolic medium

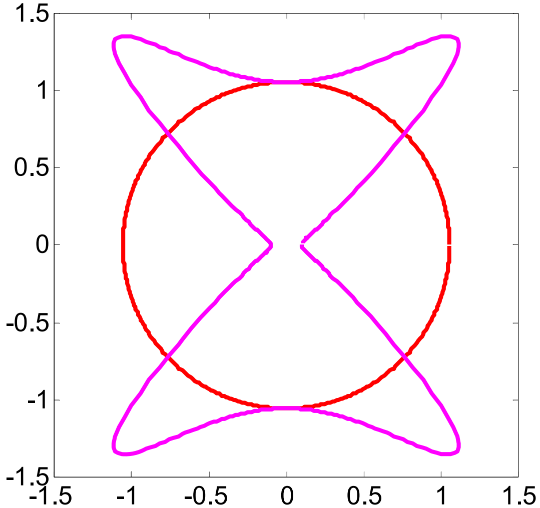
$$\overline{\varepsilon}_{eff} = \begin{pmatrix} -1.1+0.3i & 0 & 0 \\ 0 & -1.1+0.3i & 0 \\ 0 & 0 & 1+0.2i \end{pmatrix}$$

$$\overline{\varepsilon}_{eff} = \begin{pmatrix} 1+0.2i & 0 & 0 \\ 0 & 1+0.2i & 0 \\ 0 & 0 & -1.1+0.3i \end{pmatrix}$$

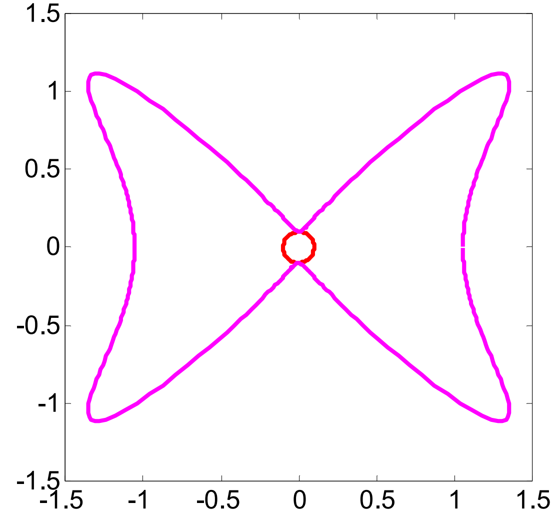


Re$\{l_x\}$, Re$\{l_z\}$





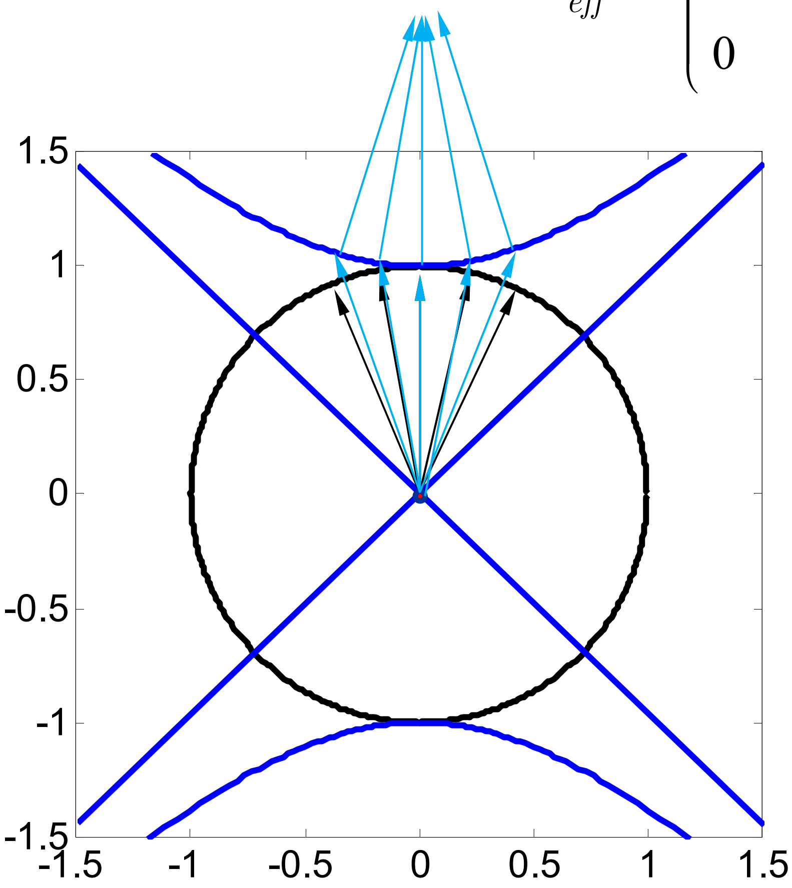
Im$\{l_x\}$, Im$\{l_z\}$

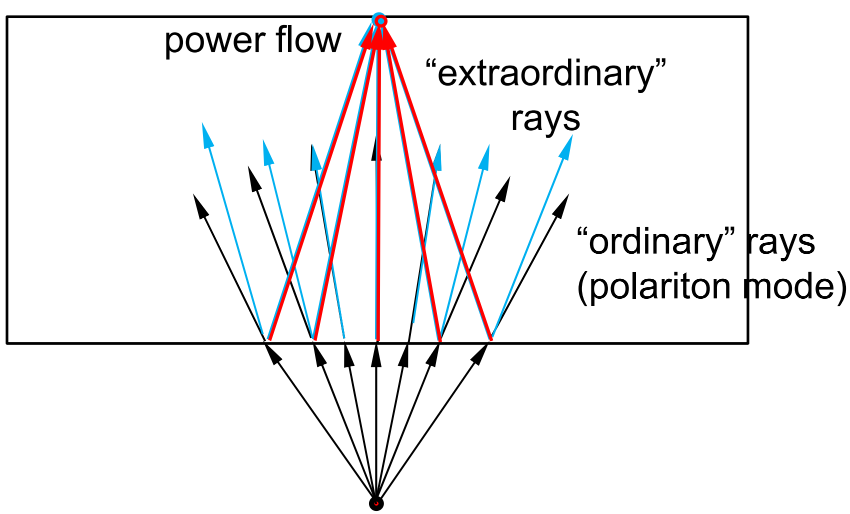


Potential application:

Imaging by a planar lens made of a metallic "nanowire" hyperbolic material

$$\bar{\boldsymbol{\varepsilon}}_{eff} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}, \quad \varepsilon_1 > 0, \quad \varepsilon_2 < 0$$





... and some others...

The END