Electric Quadrupole Radiation

Now let's to untangle the first (symmetric) piece. This will turn out to be a remarkably unpleasant job. In fact it is my nefarious and sadistic plan that it be *so* unpleasant that it properly motivates a change in approach to one that handles this nasty tensor stuff ``naturally".

We have to evaluate the integral of the symmetric piece. We get:

$$\frac{1}{2}\int[(\hat{\boldsymbol{n}}\cdot\boldsymbol{x}')\boldsymbol{J}+(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\boldsymbol{x}']d^3x' = -\frac{i\omega}{2}\int\boldsymbol{x}'(\hat{\boldsymbol{n}}\cdot\boldsymbol{x}')\rho(\boldsymbol{x}')d^3x' \qquad (11.148)$$

The steps involved are:

1. integrate by parts (working to obtain divergences of $\boldsymbol{J}$).
2. changing $\boldsymbol{\nabla}\cdot\boldsymbol{J}$ into a ρ times whatever from the continuity equation (for a harmonic source).
3. rearranging and recombining.

Don't forget the boundary condition at infinity ($\boldsymbol{J}$ and ρ have compact support)! You'll love doing this one...

The vector potential is thus:

$$\boldsymbol{A}_{\rm E2}(\boldsymbol{x}) = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int\boldsymbol{x}'(\hat{\boldsymbol{n}}\cdot\boldsymbol{x}')\rho(\boldsymbol{x}')d^3x'. \qquad (11.149)$$

Note that $\boldsymbol{x}'$ appears *twice* under the integral, and that its vector character similarly appears twice: once in $\boldsymbol{x}'$ itself and once in its projection on $\hat{\boldsymbol{n}}$. The integral is the **electric quadrupole moment** of the oscillating charge density distribution and the resulting radiation field is called an **electric quadrupole (radiation) field** or an **E2 radiation field** (for short).

To get the fields from this expression by taking its curl, and then the curl of its curl, is - ahem - most unpleasant. *Jackson* wimps out! Actually, taking the curls is no more difficult than it was for the magnetic term, but untangling the integrals with the result is, because of the tensor forms that appear. Consequently we too will wimp out (in the comforting knowledge that we will shortly do this *right* and *not* wimp out to arbitrary order in a precise decomposition) and will restrict our attention to the far zone.

There we need only consider the lowest order surviving term, which always comes from the curl of the exponential times the rest:

$$\boldsymbol{B} = ik(\hat{\boldsymbol{n}}\times\boldsymbol{A}) \qquad (11.150)$$

$$\boldsymbol{E} = ik\sqrt{\frac{\mu_0}{\epsilon_0}}(\hat{\boldsymbol{n}} \times \boldsymbol{A}) \times \hat{\boldsymbol{n}}. \tag{11.151}$$

If we keep only the lowest order terms of *this* we get

$$\boldsymbol{B} = -\frac{ick^2\mu_0}{8\pi}\frac{e^{ikr}}{r}\int(\hat{\boldsymbol{n}} \times \boldsymbol{x}')(\hat{\boldsymbol{n}} \cdot \boldsymbol{x}')\rho(\boldsymbol{x}')d^3x'. \tag{11.152}$$

If we recall (from the beginning of Chapter 4) the discussion and definition of multipole moments, in particular the **quadrupole moment tensor**

$$Q_{\alpha\beta} = \int(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\rho(\boldsymbol{x}')d^3x' \tag{11.153}$$

whose various components can be related to the five spherical harmonics with $\ell = 2$ (!) we can simplify matters. We can write the one messy integral in terms of another:

$$\hat{\boldsymbol{n}} \times \int \boldsymbol{x}'(\hat{\boldsymbol{n}} \cdot \boldsymbol{x}')\rho(\boldsymbol{x}')d^3x' = \frac{1}{3}\hat{\boldsymbol{n}} \times \boldsymbol{Q}(\hat{\boldsymbol{n}}) \tag{11.154}$$

where

$$\boldsymbol{Q}(\hat{\boldsymbol{n}}) = \sum_\beta Q_{\alpha\beta}n_\beta\hat{\boldsymbol{x}}_\beta. \tag{11.155}$$

Note that the ``vector'' $\boldsymbol{Q}(\hat{\boldsymbol{n}})$ (and hence the fields) depends in both the magnitude and direction on the direction to the point of observation **n** as well as the properties of the source. With these definitions,

$$\boldsymbol{B} = -\frac{ick^3\mu_0}{24\pi}\frac{e^{ikr}}{r}(\hat{\boldsymbol{n}} \times \boldsymbol{Q}(\hat{\boldsymbol{n}})) \tag{11.156}$$

which looks (except for the peculiar form of **Q**) much like the E1 magnetic field. It is transverse. The electric field is obtained by appending $\times \mathbf{n}$ and is also transverse. Following exactly the same algebraic procedure as before, we find from

$$\boldsymbol{S} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{11.157}$$

and computing the flux of the Poynting vector through a sphere of radius r as a function of angle that the angular power distribution is:

$$\frac{dP}{d\Omega} = \frac{c^2}{1152\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}k^6 \mid (\hat{\boldsymbol{n}} \times \boldsymbol{Q}(\hat{\boldsymbol{n}})) \times \hat{\boldsymbol{n}} \mid^2 \tag{11.158}$$

The angular distribution is too complicated to play with further unless you need to calculate it, in which case you will have to work it out. The total power can be calculated in a ``straightforward'' way (to quote Jackson). First one changes the cross product to dot products using the second relation on the front cover and squares it. One then writes out the result in tensor components. One can then perform the angular integrals of the products of the components of the **n** (which is straightforward). Finally one term in the resulting expression goes away because $Q_{\alpha\beta}$ is traceless. The result is

$$P = \frac{c^2 k^6}{1440\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{\alpha,\beta} \mid Q_{\alpha\beta} \mid^2 \qquad (11.159)$$

(note k^6 frequency dependence). For the numerologists among you, note that there is almost certainly some sort of cosmic significance in the 1440 in the denominator as this is the number of seconds in a day.

Just kidding.

For certain symmetric distributions of charge the general quadrupole moment tensor simplifies still further. A typical case of this occurs when there is an additional, e. g. azimuthal symmetry such as an oscillating spheroidal distribution of charge. In this case, the off-diagonal components of $Q_{\alpha\beta}$ vanish and only two of the remaining three are independent. We can write

$$Q_{33} = Q_0, \quad Q_{11} = Q_{22} = -\frac{1}{2} Q_0 \qquad (11.160)$$

and the angular distribution of radiated power is

$$\frac{dp}{d\Omega} = \frac{c^2 k^6}{512\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}} Q_0^2 \sin^2\theta \cos^2\theta \qquad (11.161)$$

which is a four-lobed radiation pattern characteristic of azimuthally symmetric sources. In this case it really is straightforward to integrate over the entire solid angle (or do the sum in the expression above) and show that:

$$P = \frac{c^2 k^6}{960\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} Q_0^2. \qquad (11.162)$$

At this point it should be clear that we are off on the wrong track. To quote Jackson:

> The labor involved in manipulating higher terms in (the multipolar expansion of $\mathbf{A}(\underline{x})$) becomes increasingly prohibitive as the expansion is extended beyond the electric quadrupole terms.

Some would say that we should have quit after the electric dipole or magnetic dipole.

The problem has several roots. First, in the second and all succeeding terms in the expansion as written, the magnetic and electric terms are all mixed up and of different tensorial character. This means that we have to project out the particular parts we want, which is not all that easy even in the simplest cases. Second, this approach is useful only when the wavelength is long relative to the source ($kd << 1$) which is not (always) physical for radio antennae. Third, what we have done is algebraically inefficient; we keep having to do the same algebra over and over again and it gets no easier.

Understanding the problem points out the way to solve it. We must start again at the level of the Green's function expansion, but this time we must construct a generalized *tensorial* multipolar expansion to use in the integral equation. After that, we must do ``once and for all'' the necessary curl and divergence algebra, and classify the resulting parts according to their formal transformation properties. Finally, we will reassemble the solution in the new **vector** multipoles and glory in its formal simplicity. Of course, the catch is that it is a lot of work at first. The payoff is that it is *general* and *systematically extendable* to all orders.

As we do this, I'm leaving you to work out the various example problems in Jackson (e.g. section J9.4, 9.5) on your own. We've already covered most of J9.6 but we have to do a bit more review of the angular part of the Laplace operator, which we largely skipped before. This will turn out to be key as we develop Multipolar Radiation Fields *properly*.

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